Hierarchical Laplacian

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joint work with A. Grigory'an

Hierarchical Laplacian

The data

- (X, d) - ultra-metric space (locally compact, separable)
- m - Radon measure on X
- The choice function $C\colon \mathbb{B}(X) \to (0, \infty)$ such that

$$\boxed{\lambda(B) := \sum_{T\colon B\subset T} C(T) < \infty.}$$

The Laplacian

- $P_B f := \frac{1}{m(B)} \int_B f \, \mathrm{d}m$
- $Lf(x) := -\sum_{B\colon x\in B} C(B)\Big(P_B f - f(x)\Big)$ - pointwise

Eigenfunctions & Eigenvalues

We define

- $f_V := \frac{\mathbf{1}_V}{m(V)} - \frac{\mathbf{1}_{V'}}{m(V')}$ for $V \in \mathbb{B}(X)$
- $\mathcal{H}(W) := \mathrm{span}\{f_V \ : \ V' = W\}$

Properties

(1) $Lf_V = \lambda(V')f_V$

(2) $W' \neq W \Rightarrow \mathcal{H}(W') \perp \mathcal{H}(W)$

(3) $\bigoplus \mathcal{H}(W) = L^2(X, m)$

Conclusion: $(L, \mathcal{D})$ is essentiailly self-adjoint.

Markov semigroup

$$e^{-tL}f(x) = \int_X h(t,x,y)f(y)\mathrm{d}m(y)$$

Modified distance

$$d_*(x,y) := \begin{cases} \frac{1}{\lambda(x\wedge y)} & x \neq y \\ 0 & \text{otherwise} \end{cases}$$

- In particular, eigenvalues satisfy $\lambda(B) = \frac{1}{\mathrm{diam}_*(B)}$

Spectral function

$$N(x,\tau) = \frac{1}{m(B^*_{1/\tau}(x))}$$

Heat kernel

$$h(t,x,y) = t\int_0^{1/d_*(x,y)} e^{-t\tau} N(x,\tau)\,\mathrm{d}\tau$$

Heat kernel bounds

Assume that N is doubling, then

$$h(t,x,y) \asymp \frac{t}{t+d_*(x,y)} \cdot N\left(x, \frac{1}{t+d_*(x,y)}\right)$$

Moment estimates $E\left(\left(d_*(x,\mathcal{X}_t)\right)^\gamma : \mathcal{X}_0 = x\right) \asymp$

- $\frac{t^\gamma}{1-\gamma}$ when X is perfect and non-compact
- $\frac{1}{1-\gamma}\min t, t^\gamma$ when X is discrete
- t $(\gamma > 1)$; $t(\log\frac{1}{t}+1)$ $(\gamma = 1)$; $t^\gamma(\gamma < 1)$ when X is compact

Examples

p-adic fractional derivative

$$(D^\alpha f)^\wedge(\xi) = \|\xi\|_p^\alpha \hat{f}(\xi)$$

(1) $\mathrm{Spec}(D^\alpha) = \{p^{\alpha k} : k \in \mathbb{Z}\}$

(2) $D^\alpha f(x) = \sum_{B:\, x \in B} C(B)\Big(P_B f - f(x)\Big)$,
where $C(p^k\mathbb{Z}_p) = p^{\alpha k}(1 - p^{-\alpha})$

(3) $p(t,x,y) \asymp \frac{t}{(t^{1/\alpha} + \|x-y\|_p)^{1+\alpha}}$

(4) $E(\|\mathcal{X}_t\|_p^\gamma : \mathcal{X}_0 = 0) \asymp \frac{t^{\gamma/\alpha}}{\alpha - \gamma}$

Vladimirov Laplacian

$$D^\alpha f(x) = \sum_{i=1}^d D_{x_i}^{\alpha_i} f(x),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{Q}_p^d$.

- $\|x\|_{p,\alpha} := \max_i \{\|x\|_p^{\alpha_i}\}$

$$(D^\alpha f, f) = \int_{\mathbb{Q}_p^d \times \mathbb{Q}_p^d} \Big(f(x) - f(y)\Big) J_\alpha(\mathrm{d}x, \mathrm{d}y)$$

where J_α is a singular measure.

- Transient case: $A = \sum_{i=1}^d \frac{1}{\alpha_i} > 1$
- Green function estimates (when $\alpha_1 = \ldots = \alpha_d = \beta$, $\frac{d-1}{2} < \beta < d$)

$$G(x,y) \asymp \|x-y\|_{p,\alpha}^{1-A}$$