p-Adic dynamical systems of Chebyshev polynomials

Bertin DIARRA , Djeidi SYLLA

March 21, 2013

Abstract

We study the behaviour of the iterates of the Chebyshev polynomials of the first kind in p-adic fields. In particular, we determine in the field of complex p-adic numbers for $p > 2$, the periodic points of the p-th Chebyshev polynomial of the first kind. These periodic points are attractive points. We describe their basin of attraction. The classification of finite fields extensions of the field of p-adic numbers $\mathbb{Q}_p$, enables one to locate precisely, for any integer $\nu \geq 1$, the ν-periodic points of T_p: they are simple and the nonzero ones lie in the unit circle of the unramified extension of $\mathbb{Q}_p, (p > 2)$ of degree ν. This generalizes a result, stated by M. Zuber in his PhD thesis, giving the fixed points of T_p in the field $\mathbb{Q}_p, (p > 2)$.

1 Classical formulas for Chebyshev polynomials

Let $i = \sqrt{-1}$ and $\mathbb{Q}[i]$ the quadratic field over the tfield of the rational numbers. Let $\mathbb{Q}[i][[\theta]]$ (resp.$\mathbb{Q}[i]((\theta))$ be the algebra (resp. the field) of formal power series (resp. formal Laurent series) with indeterminate θ and cœfficients in $\mathbb{Q}[i]$.

Let us consider the following formal trigonometric series, elements of $\mathbb{Q}[i][[\theta]]$:

$$\exp(i\theta) = \sum_{k\geq 0} \frac{i^k}{k!}\theta^k$$

$$\cos(\theta) = \frac{\exp(i\theta) + \exp(-i\theta)}{2} = \sum_{k\geq 0} \frac{(-1)^k}{(2k)!}\theta^{2k}$$

$$\sin(\theta) = \frac{\exp(i\theta) - \exp(-i\theta)}{2i} = \sum_{k\geq 1} \frac{(-1)^{k-1}}{(2k+1)!}\theta^{2k+1}$$

One has $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$. In the algebra of formal power series in two variables $\mathbb{Q}[i][[\theta, \theta']]$, from the fact that $\exp(i(\theta + \theta')) = \exp(i\theta) + \exp(i\theta')$ one deduces the usual addition and substraction formulas for the formal trigonometric series, $\cos(\theta)^2 + \sin(\theta)^2 = 1$, etc...

There exists a sequence of polynomials $(T_n)_{n\geq 0}$ such that $T_n(cos\theta) = \cos(n\theta)$. The polynomial T_n is called the *n-th Chebyshev polynomial of the first kind.*

If m and n are positive integers one has $T_m(T_n(\cos(\theta))) = T_m(\cos(n\theta)) = \cos(mn\theta) = T_{mn}(\cos(\theta))$. On the other hand since $\cos((n+1)\theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$ and $\cos((n-1)\theta) = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta)$, one sees that $T_{n+1}(\cos(\theta)) + T_{n-1}(\cos(\theta)) = T_1(\cos(\theta))T_n(\cos(\theta))$. As a consequence, one has

Lemma 1 *The sequence of Chebyshev polynomials of the first kind satisfies the following properties:*
−(1)− $T_0 = 1, T_1(x) = x$
−(2)− $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ \forall n \geq 1$
−(3)− $T_m \circ T_n = T_{mn} = T_n \circ T_m$

A consequence of the property (3) is that the sequence $(T_n)_{n\geq 0}$ is a commutative monoid with respect to the operation of composition, the identity element being $T_1 = x$ and $T_n^{\circ k} = T_{n^k},\ \forall n, k$.

Corollary 2 *The polynomial T_n is of degree n, whose cœfficients are integers with its leading cœfficent equal to 2^{n-1}*

Differentiating the relation $T_n(cos\theta) = \cos(n\theta)$, one obtains $\frac{d}{d\theta}T_n(\cos(\theta)) = -\sin(\theta)T_n'(cos(\theta)) = -n\sin(n\theta)$. Then $T_n'(cos\theta)) = n\frac{\sin(n\theta)}{\sin(\theta)}$.
The sequence of *Chebyshev polynomials of the second kind* is the sequence of polynomials U_n such that $U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$. Since $\sin((n+1)\theta) + \sin((n-1)\theta) = 2\cos(\theta)\sin(n\theta)$, one sees that $U_n(\cos(\theta)) + U_{n-2}(\cos(\theta)) = 2\cos(\theta)U_{n-1}(\cos(\theta))$.

Lemma 3 *The sequence of Chebyshev polynomials of the second kind satisfies the following properties:*
−(1)− $U_0 = 1, U_1(x) = 2x$.
−(2)− $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),\ \forall n \geq 1$. *The degree of U_n is equal to n.*
−(3)− $T_n'(x) = nU_{n-1}(x)$.

Let us do the change of variable by putting $\exp(i\theta) = y$. Then formally, the TChebyshev polynomials of first and second kinds are also given by substitution of the fraction $\frac{y+y^{-1}}{2}$:

$$T_n\left(\frac{y+y^{-1}}{2}\right) = \frac{y^n + y^{-n}}{2}$$

$$U_n\left(\frac{y+y^{-1}}{2}\right) = \frac{y^{n+1} - y^{-n-1}}{y - y^{-1}}$$

One consequence is that for any integer $n \geq 0$, one has:

$$T_n(-x) = (-1)^n T_n(x) \text{ and } U_n(-x) = (-1)^n U_n(x).$$

Let us do another change of variable $\dfrac{y+y^{-1}}{2} = x$, that is $y^2+1 = 2xy \iff (y-x)^2 = -(1-x^2) = i^2(1-x^2)$. The square roots of $(1-x^2)$ exist in the ring of formal power series $\mathbb{Q}[i][[x]]$, with one taken to be $\sqrt{1-x^2} = \sum_{\ell \geq 0}(-1)^\ell \binom{\frac{1}{2}}{\ell} x^{2\ell}$. Therefore $y = x \pm i\sqrt{1-x^2}$. Putting $y = x + i\sqrt{1-x^2}$, one has $y^{-1} = x - i\sqrt{1-x^2}$ and

$$T_n(x) = \frac{1}{2}\left(x+i\sqrt{1-x^2}\right)^n + \frac{1}{2}\left(x-i\sqrt{1-x^2}\right)^n$$

Since $\left(x+i\sqrt{1-x^2}\right)^n = \sum_{m=0}^{n}\binom{n}{m} i^m x^{n-m}(1-x^2)^{\frac{m}{2}} = \sum_{2m\leq n}\sum_{m=0}^{n}\binom{n}{2m}(-1)^m x^{n-2m}(1-x^2)^m +$
$i\sum_{2m+1\leq n}\binom{n}{m}(-1)^m x^{n-2m-1}(1-x^2)^{\frac{2m+1}{2}}$ and
$\left(x-i\sqrt{1-x^2}\right)^n = \sum_{2m\leq n}\binom{n}{2m}(-1)^m x^{n-2m}(1-x^2)^m$
$-\,i\sum_{2m+1\leq n}\binom{n}{2m+1}(-1)^m x^{n-2m-1}(1-x^2)^{\frac{2m+1}{2}}$, one obtains

$T_n(x) = \sum_{2m\leq n}\binom{n}{2m}(-1)^m x^{n-2m}(1-x^2)^m.$

On the other hand $(1-x^2)^m = \sum_{k+\ell=m}\binom{m}{k}(-1)^\ell x^{2\ell}$.

Then $T_n(x) = \sum_{2k+2\ell\leq n}\binom{n}{2k+2\ell}(-1)^{k+2\ell}\binom{k+\ell}{k}x^{n-2k-2\ell}x^{2\ell} =$
$= \sum_{2k\leq n}(-1)^k\sum_{2\ell\leq n}\binom{n}{2k+2\ell}\binom{k+\ell}{k}x^{n-2k}$. That is

$$T_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\sum_{\ell=0}^{[\frac{n}{2}]}\binom{n}{2k+2\ell}\binom{k+\ell}{k}x^{n-2k}.$$

By differentiating the relation $\sin(\theta)T_n'(cos(\theta)) = n\sin(n\theta)$, one obtains
$\cos(\theta)T_n'(cos(\theta)) - \sin(\theta)^2T_n''(cos(\theta)) = n^2cos(n\theta)$. Hence T_n satisfies the differential equation

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$$

Hence, setting $T_n(x) = \sum_{k=0}^{[\frac{n}{2}]}a_{n,k}x^{n-2k}$, one sees that
$(n-2k+2)(n-2k+1)a_{n,k-1} + 4k(n-k)a_{n,k} = 0$.
And by telescoping, one obtains $a_{n,k} = (-1)^k 2^{n-2k-1}\dfrac{n}{n-k}\binom{n-k}{k}$. Therefore

$$T_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \qquad (1)$$

An obvious consequence is that for any $k \leq \left[\frac{n}{2}\right]$, one has the combinatorics equality

$$\sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m+2k} \binom{m+k}{m} = 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k}.$$

Since $T_n'(x) = nU_{n-1}(x)$, one has

$U_{n-1}(x) = \frac{1}{n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k 2^{n-2k-1} (n-2k) \frac{n}{n-k} \binom{n-k}{k} x^{n-2k-1}$, which turns to be

$U_{n-1}(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k 2^{n-2k-1} \binom{n-2k}{k} x^{n-2k-1}$. And

$$U_n(x) = \sum_{k=0}^{[\frac{n+1}{2}]} (-1)^k 2^{n-2k} \binom{n-2k+1}{k} x^{n-2k}.$$

2 Fixed points of the Chebyshev polynomial T_n

Let K be an algebraically closed field of characteristic 0.

An element $x \in K$ is called a fixed point of T_n if $T_n(x) = x$. For $T_0 = 1$, the only fixed point is 1 and for $T_1 = x$, any element of K is a fixed point. Hence in the sequel we assume that $n \geq 2$.

Set $x = \frac{y+y^{-1}}{2}$, then $T_n(x) = T_n\left(\frac{y+y^{-1}}{2}\right) = \frac{y^n + y^{-n}}{2} = \frac{y+y^{-1}}{2}$

$\iff y^{2n} + 1 = y^{n+1} - y^{n-1} \iff (y^{n-1} - 1)(y^{n+1} - 1) = 0 \iff y^{n-1} = 1$ or $y^{n+1} = 1$.

Hence $y = \zeta$ is a $(n-1)$-th root of unity or $y = \eta$ is a $(n+1)$-th root of unity, and

$x = \frac{\zeta + \zeta^{-1}}{2}$ or $x = \frac{\eta + \eta^{-1}}{2}$.

Let us notice that $1 = \frac{y+y^{-1}}{2} \iff 2y = y^2 + 1 \iff (y-1)^2 = 0 \iff y = 1$. It follows that $T_n(1) = 1$, that is 1 is a fixed point of T_n.

Let us notice that this induces the following combinatorics equalities :

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} = 1 = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \sum_{\ell=0}^{[\frac{n}{2}]} \binom{n}{2k+2\ell} \binom{k+\ell}{k}.$$

If $x \neq 1$ is a fixed point of T_n, we notice above that:

−†− $x = \dfrac{\zeta + \zeta^{-1}}{2}$, with $\zeta \neq 1$ a $(n-1)$-th root of unity.
Then $T_n'(x) = nU_{n-1}(x) = n\dfrac{\zeta^n - \zeta^{-n}}{\zeta - \zeta^{-1}} = n\dfrac{\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} = n$.

−††− Or $x = \dfrac{\eta + \eta^{-1}}{2}$, with $\eta \neq 1$ a $(n+1)$-th root of unity.
Then $T_n'(x) = nU_{n-1}(x) = n\dfrac{\eta^n - \eta^{-n}}{\eta - \eta^{-1}} = n\dfrac{\eta^{-1} - \eta}{\eta - \eta^{-1}} = -n$.

−†††− One has $U_{n-1}\left(\dfrac{y + y^{-1}}{2}\right) = \dfrac{y^n - y^{-n}}{y - y^{-1}} = \displaystyle\sum_{j=0}^{n-1} y^{n-2j-1}$.
It follows that $U_{n-1}(1) = \displaystyle\sum_{j=0}^{n-1} 1 = n$ and $T_n'(1) = nU_{n-1}(1) = n^2$.

One deduces from the above that the fixed points of $T_n, n \geq 2$ are simple.
Furthermore, one has the combinatorics equalities :

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k 2^{n-2k-1} \binom{n-2k}{k} = n.$$

N.B.

Let n be a positive integer ≥ 2 and ν a positive integer ≥ 1. For the n-th Chebyshev polynomial of the first kind T_n, one has $T_n^{\circ\nu} = T_{n^\nu}$. Hence the ν-periodic points of T_n are the fixed points of the polynomial T_{n^ν}.

Proposition 4 *The fixed points of the n-th Chebyshev polynomial of the first kind $T_n, n \geq 2$ in the field of complex numbers $\mathbb{C}$, are the real numbers* $\cos\left(\dfrac{2k\pi}{n-1}\right), 0 \leq k \leq \dfrac{n-1}{2}$ *and* $\cos\left(\dfrac{2\ell\pi}{n+1}\right), 0 \leq \ell \leq \dfrac{n+1}{2}$.

They are repelling fixed points.

Proof

Indeed for any positive integer $n \geq 2$, the $(n-1)$-th roots of unity in $\mathbb{C}$ are $\zeta_k = \exp\left(\dfrac{2k\pi i}{n-1}\right), 0 \leq k \leq n-1$. The fixed points of T_n associated to these $(n-1)$-th roots of unity are $x_k = \dfrac{\zeta_k + \zeta_k^{-1}}{2} = \cos\left(\dfrac{2k\pi}{n-1}\right), 0 \leq k \leq \dfrac{n-1}{2}$.

The other fixed points y_ℓ of T_n are the real parts $y_\ell = \dfrac{\eta_\ell + \eta_\ell^{-1}}{2}, 0 \leq \ell \leq \dfrac{n+1}{2}$ of the $(n+1)$-th roots of unity $\eta_\ell = \exp\left(\dfrac{2\ell\pi i}{n+1}\right)$. Then $y_\ell = \cos\left(\dfrac{2\ell\pi}{n+1}\right), 0 \leq \ell \leq \dfrac{n+1}{2}$.

For the fixed points $x_k \neq 1$, one has $T_n'(x_k) = nU_{n-1}(x_k) = n$. Then $|T_n'(x_k)| = n > 1$. In the same way for the fixed points $y_\ell \neq 1$, one has $T_n'(y_\ell) = nU_{n-1}(y_\ell) = -n$. Then $|T_n'(y_\ell)| = n > 1$.

On the other hand for the fixed point 1, one has $T'_n(1) = nU_{n-1}(1) = n^2$. Then $|T'_n(1)| = n^2 > 1$.

Hence one concludes that any fixed point x of T_n in the complex number field $\mathbb{C}$ is a real number such that $|x| \leq 1$ and is a repelling point.

In contrast, in the field of complex p-adic numbers $\mathbb{C}_p$ the fixed points cannot be repelling points.

Proposition 5 *For n a positive integer ≥ 2, the fixed points x of T_n in the complex p-adic field $\mathbb{C}_p$ are :*

−(1)− indifferrent fixed points if p does not divide n

−(2)− attractive fixed points if p divides n.

Proof

Let us remind that if $x \neq 1$ is a fixed point, then $T'_n(x) = \pm n$ and for the fixed point $1, T'_n(1) = n^2$. Then if $p \not| n$, one has for x a fixed point, one has $|T'_n(x)| = 1$ and x is an indifferent fixed point. If $p|n$, then for x a fixed point $|T'_n(x)| < 1$ and x is an attractive fixed point.

3 The p-adic dynamic of $T_p,\ p > 2$

In this section we consider a prime number $p > 2$. Let ν be an integer ≥ 1 and let us set $q = p^\nu$. According to the previous N.B., for the p-th Chebyshev polynomial of the first kind and for any positive integer r, one has

${T_p}^{\circ r} = T_{p^r}$. In particular $T_q = {T_p}^{\circ \nu}$ and the *ν-periodic points of T_p are the fixed points of T_q.*

Lemma 6 *One has in the ring of polynomials $\mathbb{Z}_p[x]$, the congruence $T_q(x) \equiv x^q (mod\ p\mathbb{Z}_p[x])$.*

Proof

One has $T_q(x) = 2^{q-1}x^q + \sum_{k=1}^{[\frac{q}{2}]} (-1)^k 2^{q-2k-1} \dfrac{q}{q-k} \dbinom{q-k}{k} x^{q-2k}$.

For $1 \leq k \leq [\frac{q}{2}] < q$, one has $v_p(k) < v_p(q) = \nu$, where v_p is the p-adic valuation, hence $v_p(q-k) = \min(v_p(k), v_p(q)) = v_p(k)$ and $v_p\left(\dfrac{q}{q-k}\right) = v_p(q) - v_p(q-k) = \nu - v_p(k) > 0$. Since the binomial cœfficients $\dbinom{q-k}{k}$ are integers and $|(-1)^k 2^{q-2k-1}| = 1$, one sees that $\left|(-1)^k 2^{q-2k-1} \dfrac{q}{q-k} \dbinom{q-k}{k}\right| \leq \left|\dfrac{q}{q-k}\right| = |p|^{\nu - v_p(k)} < 1$.

From little Fermat theorem, one deduces that $2^{q-1} \equiv 1 (\text{mod } p)$.

One then obtains the congruence of the Lemma. □

Let $\mathbb{E}_q$ be the unique unramified field extension of $\mathbb{Q}_p$ of degree ν. Its residue field is the finite field $\mathbb{F}_q$ of q elements; the dimension $[\mathbb{F}_q : \mathbb{F}_p]$ of the extension $\mathbb{F}_q | \mathbb{F}_p$ is equal ν. The field $\mathbb{E}_q$ is generated over $\mathbb{Q}_p$ by a $(q-1)$-th primitive root of unity ξ (see for instance [3]) and $\mathbb{E}_q$ contains the group of $(q-1)$-th roots of unity which in fact are the Teichmüller representative of the nonzero elements of $\mathbb{F}_q$.

Since the extension $\mathbb{E}_q|\mathbb{Q}_p$ is unramified, its group of valuation is equal to those of $\mathbb{Q}_p$. Let $\Lambda_q = \{x \in \mathbb{E}_q \ / \ |x| \leq 1\}$ be the valuation ring of $\mathbb{E}_q$ The maximal ideal Λ_q is equal to $p\Lambda_q$, that is p is an uniformizer of $\mathbb{E}_q$.

Proposition 7 *Let p be a prime number > 2 and $q = p^\nu, \nu \geq 1$.*

Let $\xi_0 = 0$ and $(\xi_\ell)_{1\leq \ell \leq q-1}$ be the finite sequence of the $(q-1)$-th roots of unity ordered in such a way that the $p-1$ first are the $(p-1)$-th roots of unity with $\xi_\ell \equiv \ell (mod\ p), 1 \leq \ell \leq p-1$
Any fixed point of of the q-th Chebyshev polynomial belong to Λ_q. They can be ordered in the form $0 = w_0, w_1, \cdots, w_\ell, \cdots, w_{q-1}$ such that $w_\ell \equiv \xi_\ell (mod\ p), 0 \leq \ell \leq q-1$.

Proof

Since the maximal ideal of the valuation ring Λ_q is $p\Lambda_q$, the congruence in Lemma 6 can be extended in the form $T_q(x) \equiv x^q (\text{mod } p\Lambda_q[x])$, and one has $T_q(x) - x \equiv x^q - x (\text{mod } p\Lambda_q[x])$. It follows that $T'_q(x) - 1 \equiv -1 (\text{mod } p\Lambda_q[x])$. But the zeroes of the polynomial $x^q - x$ in the residue field $\Lambda_q/p\Lambda_q = \mathbb{F}_q$ are simple and are all the elements of this finite field. Applying Hensel lemma, one sees that the zeroes $w_0, w_1, \cdots, w_\ell, \cdots, w_{q-1}$ of the polynomial $T_q(x) - x$ are simple and the set of their classes $\{\overline{w}_k, 0 \leq \ell \leq q-1\}$, modulo $p\Lambda_q$ is equal to $\mathbb{F}_q$. Setting $w_0 = 0$ and ξ_ℓ the Teichmüller representative of $\overline{w}_\ell$ in Λ_q, which is a $(q-1)$-th root of unity, one has $w_\ell \equiv \xi_\ell (\text{mod } p)$.
□

An immediate consequence is that *the absolute value of any nonzero fixed point of T_q is equal to* 1

N.B. One has another proof of Proposition 7, by using the following Lemma and the fact that the fixed points of T_q can be expressed, or in the form $x = \dfrac{\xi + \xi^{-1}}{2}$, with ξ a $(q-1)$-th root of unity or in the form $y = \dfrac{\eta + \eta^{-1}}{2}$, with η a $(q+1)$-th root of unity.

Lemma 8 *Let $q = p^\nu$ be a power of the prime > 2.*
The unramified extension $\mathbb{E}_{q^2}$ of $\mathbb{Q}_p$ of degree 2ν contains $\mathbb{E}_q$ and $[\mathbb{E}_{q^2} : \mathbb{E}_q] = 2$.
The field $\mathbb{E}_{q^2}$ is generated over $\mathbb{Q}_p$ by the $(q-1)$-th and $(q+1)$-th roots of unity .
Moreover any $(q+1)$-th root of unity η in $\mathbb{E}_{q^2}$ is such that $\dfrac{\eta + \eta^{-1}}{2}$ belongs to $\mathbb{E}_q$.

We omit the proof of Lemma 8. □

Lemma 9 *Let $L_q = \{a \in \mathbb{C}_p \ / \ |a| \leq 1$ and $|a^q - a| \leq |p|^{\frac{1}{p-1}}\}$. Then $\Lambda_q \subset L_q$.*
Let m be a positive integer coprime to p.
For any $a \in L_q$ the sequence $(T_{mq^k}(a))_{k \geq 0}$ converges in $\mathbb{C}_p$ and if a belongs to Λ_q, then $\lim\limits_{k \to +\infty} T_{mq^k}(a) \in \Lambda_q$.

Proof

The polynomial congruence $T_q(x) \equiv x^q (\text{mod } p\mathbb{Z}_p[x])$, means that $T_q(x) = x^q + pr_q(x)$, with $r_q(x) \in \mathbb{Z}_p[x]$. Hence for any $t \in \mathbb{C}_p$ such that $|t| \leq 1$, one has $|r(t)| \leq 1$ and $|p||r_q(t)| \leq |p| \leq |p|^{\frac{1}{p-1}}$. Since $T_{mq^{k+1}}(t) = T_{mq^k} \circ T_q(t) = T_{mq^k}(t^q + pr_q(t))$; applying the p-adic mean value theorem (cf [7] or [8]), one sees that $|T_{mq^{k+1}}(t) - T_{mq^k}(t^q)| = |T_{mq^k}(t^q + pr_q(t)) - T_{mq^k}(t^q)| \leq |pr_q(t)| \| T'_{mq^k} \|$, where $\| T'_{mq^k} \|$ is the Gauss norm of the polynomial $T'_{mq^k} = mq^k U_{mq^k-1}$, and where U_{mq^k-1} is the (mq^k-1)-th Chebyshev polynomial of the second kind U_{mq^k-1} whose cœfficients are seen to be integer numbers. Then $\| T'_{mq^k} \| = |mq^k| \| U_{mq^k-1} \| \leq |q|^k$ and $|T_{mq^{k+1}}(t) - T_{mq^k}(t^q)| \leq \; |pr_q(t)||q|^k \leq |p||q|^k$.

Let $a \in L_q$, then applying again the p-adic mean value theorem, one has $|T_{mq^k}(a^q) - T_{mq^k} a)| = |T_{mq^k}(a + (a^q - a)) - T_{mq^k} a)| \leq |a^q - a| \| T'_{mq^k} \| \leq |p|^{\frac{1}{p-1}} |mq^k| = |p|^{\frac{1}{p-1}} |q^k|$.

Hence for any $a \in L_q$, one obtains:
$|T_{mq^{k+1}}(a) - T_{mq^k}(a)| = |T_{mq^k}(a^q + pr(a)) - T_{mq^k}(a^q) + T_{mq^k}(a^q) - T_{mq^k}(a)| \leq$
$\leq \max(|T_{mq^k}(a^q + pr(a)) - T_{mq^k}(a^q)|, |T_{mq^k}(a^q) - T_{mq^k}(a)|) \leq$
$\leq \max(|p|, |p|^{\frac{1}{p-1}})|q|^k = |p|^{\frac{1}{p-1}} |q|^k$.

Il follows that for any element a of L_q, one has $\lim_{k \to +\infty} |T_{mq^{k+1}}(a) - T_{mq^k}(a)| = 0$ and the sequence $(T_{mq^k}(a))_{k \geq 0}$ is a Cauchy sequence and then converges in $\mathbb{C}_p$.

Since the residue field of the ring Λ_q is the finite field $\mathbb{F}_q$, any $a \in \Lambda_q$ is such that
$a^q \equiv a (\text{mod } p\Lambda_q)$, then $|a^q - a| < |p| \leq |p|^{\frac{1}{p-1}}$, that is a belongs to L_q. Since the cœfficients of the polynomials are integer numberss, for any $a \in \Lambda_q$, one has $T_{mq^k}(a) \in \Lambda_q$ and $\lim_{k \to +\infty} T_{mq^k}(a) \in \Lambda_q$.
$\square$

Let us set $\varphi_m(a) = \lim_{k \to +\infty} T_{mq^k}(a)$, for $a \in L_q$ and m a positive integer coprime to p.
One has $\varphi_m(a) = \lim_{k \to +\infty} T_m \circ T_{q^k}(a) = T_m \left(\lim_{k \to +\infty} T_{q^k}(a) \right) = T_m(\varphi_1(a))$. On the other hand $\varphi_1(a) = \lim_{k \to +\infty} T_q \circ T_{q^{k-1}}(a) = T_q \left(\lim_{k \to +\infty} T_{q^{k-1}}(a) \right) = T_q(\varphi_1(a))$.
Then $\varphi_1(a)$ is a fixed point of T_q.

The closed discs in $\mathbb{C}_p$ (resp. $\mathbb{E}_q$) will be denoted by $D^+(a, r)$ (resp. $D^+_q(a, r)$) and the open discs by $D^-(a, r)$ (resp. $D^-_q(a, r)$).
Let us put $|p|^{\frac{1}{p-1}} = \rho_p$.

Remark 10 *Let $\xi_0 = 0$ and $\xi_1, \xi_1, \cdots, \xi_{q-1}$ be the $(q-1)$-th roots of unity.*
Then $D^+(\xi_\ell, \rho_p) \subset L_q$. In fact $L_q = \bigsqcup_{0 \leq \ell \leq q-1} D^+(\xi_\ell, \rho_p)$.

Proof

Indeed, if $a \in D^+(\xi_\ell, \rho_p)$, since $\xi_\ell^q = \xi_\ell$, for $1 \le \ell \le q-1$, one has $a^q - a = a^q - \xi_\ell^q + \xi_\ell^q - a = a^q - \xi_\ell^q + \xi_\ell - a$, with $|a^q - \xi_\ell^q| = |a - \xi_k| \left| \sum_{j=0}^{\ell-1} a^{q-1-j} \xi_k^j \right| \le |a - \xi_\ell|$. It follows that $|a^q - a| \le \max(|a^q - \xi_\ell^q|, |\xi_\ell - a|) = |a - \xi_k| \le \rho_p$ and $D^+(\xi_\ell, \rho_p) \subset L_q$.

For $\ell = 0$, one has $|a| = |a - 0| \le \rho_p$ and obviously $|a^q - a| = |a| \le \rho_p$.

Let $a \in L_q$, since $|a^q - a| \le \rho_p < 1$, one sees on one hand that $|a^{q^k} - a^{q^{k-1}}| \le \rho_p$ and on the other hand, since $a^q = a + c$, with $|c| < 1$, one verifies that $(a^{q^k})_{k\ge 0}$ is a Cauchy sequence. Hence, setting $\omega(a) = \lim_{k\to+\infty} a^{q^k}$, one has $\omega(a)^q = \omega(a)$ and $|\omega(a) - a| \le \rho_p$. If $\omega(a) = 0$, then a belongs to $D^+(0, \rho_p)$. Otherwise, one has $\omega(a) \ne 0$ and $\omega(a)$ is a $(q-1)$-th root of unity and equal to one of the ξ_ℓ, then $|a - \xi_\ell| \le \rho_p$, that is a belongs to $D^+(\xi_\ell, \rho_p)$.

It is readily seen that two distinct discs $D^+(\xi_\ell, \rho_p)$ has an empty intersection.

N.B.

$-\&-$ *Let us set* $L_q^- = \{a \in \mathbb{C}_p \ / \ |a^q - a| < 1\}$. *Then as above* $L_q^- = \bigsqcup_{0\le\ell\le q-1} D^-(\xi_\ell, 1)$.

The proof is as that of Remark 10

$-\&\&-$ The sets L_q and L_q^- are described in [9] for the case where $q = p$ and called lemniscate. In fact they are special case of the lemniscates that can be attached to any monic polynomial with cœfficients in an ultrametric valued field as defined in [1] -Proposition 4.8.1.

Proposition 11 *Let $w_0 = 0, w_1, \cdots, w_{q-1}$ be the fixed points of T_q, i.e. the ν-periodic points of T_p.*

Then, for $0 \le \ell \le q-1$ and for $a \in D^+(\xi_\ell, \rho_p)$, one has $\varphi_1(a) = w_\ell$.

Proof

Let $a \in D^+(\xi_\ell, \rho_p)$, we have seen above that $|a^q - a| \le \rho_p$ and that $(a^{q^k})_{k\ge 0}$ is a Cauchy sequence.

On the other hand, the polynomials T_{q^k} are such that $T_{q^k}(x) = x^{q^k} + pr_{q^k}(x)$, the cœfficients of the polynomial $r_{q^k}(x)$ being integer numbers. Hence for $a \in D^+(\xi_\ell, \rho_p)$, one has $|T_{q^k}(a) - a^{q^k}| \le |p| < \rho_p$ and $|\lim_{k\to+\infty} T_{q^k}(a) - \lim_{k\to+\infty} a^{q^k}| = |\varphi_1(a) - \omega(a)| = |\varphi_1(a) - \xi_\ell| \le \rho_p$. As w_ℓ is the only fixed point of T_q with this property, one has $\varphi_1(a) = w_\ell$. $\square$

Remark 12 *Let T_n be the n-th Chebyshev polynomial of the first kind, $n \ge 1$*

If $p > 2$ and $a \in \mathbb{C}_p$ is such that $|a| > 1$, then $\lim_{k\to+\infty} |T_{n^k}(a)| = +\infty$.

Proof Indeed since $T_{n^k}(x) = 2^{n^k-1} x^{n^k} + \sum_{j=0}^{n^k-1} a_{n^k,j} x^j$, with $a_{n^k,j} \in \mathbb{Z}$, for $|a| > 1$, one has $|a_{n^k,j}| \cdot \dfrac{|a^j|}{|2^{n^k-1}||a^{n^k}|} \le \dfrac{|a^j|}{|a^{n^k}|} < 1$, for $0 \le j \le n^k - 1$.

Hence $|T_{n^k}(a)| = |2^{n^k-1}||a^{n^k}| \cdot \left|1 + \sum_{j=0}^{n^k-1} a_{n^k,j} \frac{a^j}{2^{n^k-1}a^{n^k}}\right| = |a|^{n^k} \longrightarrow +\infty$, when $k \longrightarrow +\infty$. $\square$

Let us remind that by definition the basin of attraction of an attractive fixed point x_0 for a dynamical system associated to a function f on a topological set X is the subset $Att(x_0, f) = \{y \in X \ / \lim\limits_{k\to+\infty} f^{\circ k}(y) = x_0\}$.

Since $T'_q = qU_{q-1}$, for the fixed points w_ℓ of T_q, one has $|T'_q(w_\ell)| = |q||U_{q-1}(w_\ell)| \le |q| < 1$ and w_ℓ is an attractive fixed point of T_q.

Theorem 13 *With the same notations as above, one has $Att(w_\ell, T_q) = D^-(\xi_\ell, 1)$.*

Proof

$- \bullet -$ Let $a \in D^-(\xi_\ell, 1)$, that is $|a - \xi_\ell| < 1$. Since $\xi_\ell^q = \xi$, one has $a^q - \xi_\ell = a^q - \xi_\ell^q = (a - \xi) \sum_{j=0}^{q-1} a^{q-1-j} \xi_\ell \Longrightarrow |a^q - \xi_\ell \le |a - \xi_\ell| < 1$ and $|a^q - a| < 1$. Since $a^q = \xi_\ell + c$, with $|c| < 1$, one sees that $a^{q^k} = \xi_\ell + c_k$, with $\lim\limits_{k\to+\infty} c_k = 0$, hence $\lim\limits_{k\to+\infty} a^{q^k} = \xi_\ell$. and he sequence $(a^{q^k})_{k\ge0}$ is a Cauchy sequence. Therefore there exists k_0 such that $\forall k \ge k_0$, one has $|a^{q^{k+1}} - a^{q^k}| < \rho_p$.
It follows that $a^{q^{k_0}}$ belongs to L_q and also $a^{q^{k_0}} \in D^+(\xi_\ell, \rho_p)$.
According to Proposition 11, one has $\lim\limits_{k\to+\infty} T_{q^\ell}(a^{q^{k_0}}) = w_\ell$.
However, $T_{q^{k_0+k}}(a) = T_{q^k}(T_{q^{k_0}}(a))$ and $T_{q_0^k}(a) = a^{q^{k_0}} + pr_{k_0}(a^{q_0^k})$, with $r_{k_0} \in \mathbb{Z}_p[x)$.

Applying again the p-adic mean value theorem, one has $|T_{q^{k_0+k}}(a) - T_{q^k}(a^{q^{k_0}})| =$
$= |T_{q^k}(a^{q^{k_0}} + pr_{k_0}(a^{q_0^k})) - T_{q^k}(a^{q^{k_0}})| \le \|T'_{q^k}\||p| \le |p||q|^k$.
It follows that $\lim\limits_{k\to+\infty} T_{q^k}(a) = \lim\limits_{k\to+\infty} T_{q^{k_0+k}}(a) = \lim\limits_{k\to+\infty} T_{q^k}(T_{q^{k_0}}(a)) = w_\ell$ and $a \in Att(w_\ell, T_q)$.
Therefore $D^-(\xi_\ell, 1) \subseteq Att(w_\ell, T_q)$.

$- \bullet -$ Let $a \in Att(w_\ell, T_q)$. By definition $\lim\limits_{k\to\infty} T_q^{\circ k}(a) = w_\ell$. Reminding that $T_q^{\circ k} = T_{q^k}$, according to Remark 12, one must have $|a| \le 1$.

There exists k_0 such that $|T_{q^k}(a) - w_\ell| \le \rho_p$, $\forall k \ge k_0$. As above $|T_{q^k}(a) - a^{q^k}| \le |p|$. Since $|w_\ell - \xi_\ell| \le |p|$, one obtains $|\xi_\ell - a^{q^k}| = |w_\ell - T_{q^k}(a) + T_{q^k}(a) - w_\ell + w_\ell - \xi_\ell| \le$
$\le \max(|w_\ell - T_{q^k}(a)|, |T_{q^k}(a) - w_\ell|, |w_\ell - \xi_\ell|) \le \rho_p < 1$. However $\xi_\ell^q = \xi_\ell \Longrightarrow \xi_\ell^{q^k} = \xi_\ell$. Hence in the residue field $\widetilde{\mathbb{F}}_p$ of $\mathbb{C}_p$, one has $0 = \overline{a}^{q^k} - \overline{\xi}^{q^k} = (\overline{a} - \overline{\xi}_\ell)^{q^k} \Longrightarrow \overline{a} = \overline{\xi}_\ell$, that is $|a - \xi_\ell| < 1 \Longrightarrow Att(w_\ell, T_q) \subseteq D^-(\xi_\ell, 1)$.

In conclusion $Att(w_\ell, T_q) = D^-(\xi_\ell, 1)$. $\square$

Corollary 14 *Let $Att_q(w_\ell, T_q)$ be the set of attracting points off w_ℓ contained in the unramified field $\mathbb{E}_q$.*
Then $Att_q(w_\ell, T_q) = D_q^+(\xi_\ell, |p|)$.

Proof It suffices to notice that the unramified field $\mathbb{E}_q$ is of discrete valuation and that if $a \in \mathbb{E}_q$ is such that $|a| < 1$, then $|a| \leq |p|$. $\square$.

Remark

Let p be a prime number $\neq 2$. What we have done above is the determination of all the periodic points of the p-th Chebyshev polynomial of the first kind T_p. Since $T_p^{\circ\nu} = T_{p^\nu}$ the p^ν-th Chebyshev polynomial of the first kind, for any periodic w, taken ν such that $T_p^{\circ\nu}(w) = w$, one has that w is a fixed point T_{p^ν} and if not equal 0, is congruent modulo the maximal ideal of the valuation ring of $\mathbb{C}_p$ to a $p^\nu - 1$-th root of unity.

Summarizing, one sees that the periodic points of T_p are in a bijective correspondence with the residue field $\widetilde{\mathbb{F}}_p$ of $\mathbb{C}_p$.

4 The p-adic dynamic of $T_n, (n,p) = 1,\ p \geq 3$

Let us remind that if p is a prime number, then for any integer $m \geq 1$ and x_0 a fixed point of T_m, one has $|T_m(x_0)| = |m|$ if $x_0 \neq 1$ and $|T_m(1)| = |m|^2$, when $x_0 = 1$. If follows that if p does not divide m, then $|T_m(x_0)| = 1$ for any fixed point x_0 of T_m. In other words any fixed point of T_m in the complex p-adic field $\mathbb{C}_p$ is an indifferent point.

Let $f : D \longrightarrow D$ be an analytic map, where D is a disc of $\mathbb{C}_p$ of finite or infinite radius and let w be a ν-periodic point of f, if there exists an *open disc $D^-(w,r)$ such that for any real number $0 < r' < r$ the sphere $S(w,r') = \{x \in \mathbb{C}_p \ / \ |x - w| = r'\}$ is invariant by f^ν, one says that $D^-(w,r)$ is a Siegel disc and w a center of a Siegel disc.* The union of Siegel discs with center w is called *the Siegel disc and then of maximal radius at w.* This is the ultrametric counterpart of the Siegel disc defined in complex analysis. (see for the complex case [2] and [4] or [5] for the ultrametric case).

Assume $p \geq 3$. Let n be a positive integer ≥ 2, such that n and p are coprime. Hence for any other integer $\nu \geq 1$ one has $(n^\nu, p) = 1$. Since $T_n^{\circ\nu} = T_{n^\nu}$, the ν-periodic points are the fixed points w of T_{n^ν} that we know be of the form $w = \dfrac{\xi + \xi^{-1}}{2}$, where ξ is a $(n^\nu - 1)-$th root of unity or of the form $w = \dfrac{\eta + \eta^{-1}}{2}$, where η is a $(n^\nu + 1)-$th root of unity.

Hence if w is a ν-periodic point of T_n in $\mathbb{C}_p$, one has $|w| \leq 1$. Moreover, if $\xi^2 + 1 \not\equiv 0 (\text{mod } \mathcal{M}_p)$, (resp. $\eta^2 + 1 \not\equiv 0(\text{mod } \mathcal{M}_p)$, where $\mathcal{M}_p$ is the maximal ideal of the valuation ring $\mathbb{A}_p$ of $\mathbb{C}_p$, then $|w| = 1$. Otherwise $|w| < 1$ and reducing modulo $\mathcal{M}_p$, one has in the residue field $\mathbb{A}_p/\mathcal{M}_p = \widetilde{\mathbb{F}}_p$ that $\overline{\zeta}^2 + 1 = 0$ (resp. $\overline{\eta}^2 + 1 = 0$). Applying Hensel lemma, one sees that the root of unity is a square root of -1 in $\mathbb{C}_p$ and then $w = 0$.

Let $T_{n^\nu}(x) = T_{n^\nu}(w) + \displaystyle\sum_{j=1}^{n^\nu} \frac{T_{n^\nu}^{(j)}(w)}{j!}(x - w)^j$ be the Taylor expansion near w, where $T_{n^\nu}^{(j)}$ is the j-th derivative of T_{n^ν}.

However, $T_{n^\nu}(x) = \displaystyle\sum_{k=0}^{[\frac{n^\nu}{2}]} (-1)^k 2^{n^\nu - 2k - 1} \frac{n^\nu}{n^\nu - k} \binom{n^\nu - k}{k} x^{n^\nu - 2k}$.

It is then readily seen that for $0 \leq j \leq n$, one has

$$\frac{T_{n^\nu}^{(j)}(x)}{j!} = \sum_{k=0}^{[\frac{n^\nu}{2}]}(-1)^k 2^{n^\nu-2k-1}\binom{n^\nu-2k}{j}\frac{n^\nu}{n^\nu-k}\binom{n^\nu-k}{k}x^{n^\nu-2k-j} =$$

$$= \sum_{k=0}^{[\frac{n^\nu-j}{2}]}(-1)^k 2^{n^\nu-2k-1}\frac{n^\nu}{n^\nu-k}\binom{n^\nu-2k}{j}\binom{n^\nu-k}{k}x^{n^\nu-2k-j}.$$

The cœfficients of the polynomials $\frac{T_{n^\nu}^{(j)}(x)}{j!}$ are integer numbers.

Hence, since $|w| \leq 1$,one sees that

$$\left|\frac{T_{n^\nu}^{(j)}(w)}{j!}\right| = \left|\sum_{k=0}^{[\frac{n^\nu-j}{2}]}(-1)^k 2^{n^\nu-2k-1}\frac{n^\nu}{n^\nu-k}\binom{n^\nu-2k}{j}\binom{n^\nu-k}{k}w^{n^\nu-2k-j}\right| \leq$$

$$\leq \max_{0\leq k\leq[\frac{n^\nu-j}{2}]}|w|^{n^\nu-2k-j} \leq 1.$$

Proposition 15 *Let p be a prime number different from 2 and n be an integer ≥ 2 such that p does not divide n.*
Then any periodic point w of T_n is an indifferent point, with absolute value ≤ 1 and the Siegel disc around w is the disc $D^-(w,1)$.

Proof

Let w be a periodic point of T_n of periodic ν. Then $T_{n^\nu}(w) = w$ and $T_{n^\nu}(x) - w =$

$$= T_{n^\nu}(x) - T_{n^\nu}(w) = \sum_{j=1}^{n^\nu}\frac{T_{n^\nu}^{(j)}(w)}{j!}(x-w)^j = (x-w)\sum_{j=1}^{n^\nu}\frac{T_{n^\nu}^{(j)}(w)}{j!}(x-w)^{j-1} =$$

$$= (x-w)\left(T_{n^\nu}(w) + \sum_{j=2}^{n^\nu}\frac{T_{n^\nu}^{(j)}(w)}{j!}(x-w)^{j-1}\right).$$

But $\left|\sum_{j=2}^{n^\nu}\frac{T_{n^\nu}^{(j)}(w)}{j!}(x-w)^{j-1}\right| \leq \max_{2\leq j\leq n^\nu}\left|\frac{T_{n^\nu}^{(j)}(w)}{j!}\right||x-w|^{j-1} \leq \max_{2\leq j\leq n^\nu}|x-w|^{j-1}$.

Then if $|x-w| < 1$, one has $\left|\sum_{j=2}^{n^\nu}\frac{T_{n^\nu}^{(j)}(w)}{j!}(x-w)^{j-1}\right| \leq \max_{2\leq j\leq n^\nu}|x-w|^{j-1} < 1$.

Since $|T'_{n^\nu}(w)| = 1$, for $|x-w| < 1$, one obtains $|T_{n^\nu}(x) - w| = |x-w|$.

Let $0 < r' < r < 1$ be two real numbers, elements of $|\mathbb{C}_p| \setminus \{0\}$.

If $|x-w| = r'$, then $|T_{n^\nu}(x) - w| = |x-w| = r'$, it follows that the sphere $S(w,r')$ is invariant by T_{n^ν}.

This shows that $D^-(w,1)$ is the Siegel disc around w.

N.B. If the integer numbers n and ν are such that $n^\nu - 1 = p^\mu$ a power of p, then if ζ is a primitive $n^\nu - 1$-th root of unity the field $\mathbb{Q}_p[\zeta]$ is a totally ramified extension of $\mathbb{Q}_p$. It follows that this field has the same residue field $\mathbb{F}_p$ as $\mathbb{Q}_p$. If $n^\nu - 1 = p^k,\ k \geq 2$ there are distinct ν-periodic points of T_n in $\left\{\frac{\xi+\xi^{-1}}{2} \neq 0, \xi^{n^\nu-1} = 1\right\}$ whose residue classes are equal in $\mathbb{F}_p$.

Scholie

−†− Above we have supposed that the prime number p is different from 2 without giving any explanation. But let us consider the expansion of the Chebyshev polynomials:

$$T_{2m}(x) = \sum_{k=0}^{m} (-1)^k 2^{2(m-k)-1} \frac{2m}{2m-k} \binom{2m-k}{k} x^{2(m-k)} =$$

$$= 2^{2m-1}x^{2m} + \sum_{k=1}^{m-1} (-1)^k \sum_{k=0}^{m} (-1)^k 2^{2(m-k)-1} \frac{2m}{2m-k} \binom{2m-k}{k} x^{2(m-k)} + (-1)^m.$$

It follows that in the polynomial ring $\mathbb{Z}[x]$, one has the congruence

$$T_{2m}(x) \equiv 1 (\text{mod } 2\mathbb{Z}[x])$$

Also $T_{2m+1} = 2^{2m}x^{2m+1} + \sum_{k=1}^{m-1} (-1)^k \sum_{k=0}^{m} (-1)^k 2^{2(m-k)} \frac{2m+1}{2m-k+1} \binom{2m-k+1}{k} x^{2(m-k)+1} + (-1)^m(2m+1)x$. Therefore

$$T_{2m+1}(x) \equiv x (\text{mod } 2\mathbb{Z}[x])$$

Hence the method used to locate periodic points of the polynomial T_p cannot be applied when $p = 2$.

Nevertheless if w in a fixed point of the Chebyshev polynomial T_n, one has in the field of 2-adic complex numbers $\mathbb{C}_2$ that $|T_n'(w)|_2 = |n|_2$ if $w \neq 1$ and $|T_n'(1)|_2 = |n^2|_2$. Then if n is even $|T_n'(w)| < 1$ and any fixed point w of T_n is an attractive point in $\mathbb{C}_2$. On the other hand, if n is odd, one has $|T_n'(w)| = 1$ and the fixed point w of T_n is indifferent. By the same way, any periodic point of T_n is an attractive point if n is even and an indifferent point if n is odd.

The fixed points of the second Chebyshev polynomial $T_2(x) = 2x^2 - 1$ are 1 and $-\frac{1}{2}$. Hence in contrast with the cas $p \neq 2$, one has $\left|-\frac{1}{2}\right|_2 = 2 > 1$.

However for any positive integer $\nu \geq 1$, the ν-periodic points in $\mathbb{C}_2$ lie in the ν-dimensional unramified extension $\mathbb{E}_{2^\nu}$ of $\mathbb{Q}_2$. Indeed one can apply Lemma 8 and the Nota Bene before it to prove that, since $T_2^{\circ\nu} = T_{2^\nu}$, the ν-periodic points are of the form $w = \frac{\xi + \xi^{-1}}{2}$ where ξ is a $(2^\nu - 1)$-th root of unity or $w = \frac{\eta + \eta^{-1}}{2}$ where η is a $(2^\nu + 1)$-th root of unity.

For $\nu = 2$, since $2^2 - 1 = 3$ the 3-th roots of unity (cubic roots of unity) in $\mathbb{C}_2$ are 1 and the roots of the polynomial $x^2 + x + 1$ that we write in the forms $j = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$ and $j^2 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. The corresponding unramified field extension of dimension 2 over $\mathbb{Q}_2$ is $\mathbb{E}_4 = \mathbb{Q}_2[j] = \mathbb{Q}_2[\sqrt{-3}]$

For $2^2 + 1 = 5$ the 5-th roots of unity are 1 and the roots of the polynomial $x^4 + x^3 + x^2 + x + 1$. A classical procedure of the resolution of this quartic equation by setting $u = x + \frac{1}{x}$ yields to the auxillary equation $u^2 + u - 1 = 0$ which has two solutions $u_\pm = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Notice that since $5 \equiv -3 (\text{mod } 8)$, one has $\sqrt{5} = \delta\sqrt{-3} \in \mathbb{E}_4$, with δ the square root of $1 - 3^{-1}8$ which belongs to $\mathbb{Q}_2$ and $\mathbb{E}_4 = \mathbb{Q}_2[\sqrt{5}]$. Moreover $x^2 - u_\pm x + 1 = 0 \Longrightarrow 2x^2 + (1+\sqrt{5})x + 2 = 0$, or $2x^2 + (1-\sqrt{5})x + 2 = 0$.

Then $\left(x+\frac{1+\sqrt{5}}{4}\right)^2=\frac{2\sqrt{5}-10}{16}$ or $\left(x+\frac{1-\sqrt{5}}{4}\right)^2=\frac{2\sqrt{5}-10}{16}$, with $\frac{2\sqrt{5}-10}{16}\in\mathbb{E}_4$. Let us denote by $\theta\in\mathbb{C}_2$ a root of the polynomial $X^2-(2\sqrt{5}-10)\in\mathbb{E}_4[X]$, one obtains the 5-th roots of unity in the form $\eta_1=-\frac{1+\sqrt{5}}{4}+\frac{\theta}{4}$, $\eta_2=-\frac{1+\sqrt{5}}{4}-\frac{\theta}{4}$, with $\eta_1^{-1}=\eta_2$ and
$\eta_3=\frac{\sqrt{5}-1}{4}+\frac{\theta}{4}$, $\eta_4=\frac{\sqrt{5}-1}{4}-\frac{\theta}{4}$, with $\eta_3^{-1}=\eta_4$. From these 5-th roots of unity in $\mathbb{C}_2$, one obtains the 2-periodic points $w_3=-\frac{1+\sqrt{5}}{4}$ and $w_4=\frac{\sqrt{5}-1}{4}$ that belongs to $\mathbb{E}_4$. The other 2-periodic points of T_2 are the fixed points 1 and $-\frac{1}{2}$. The 2-periodic points w_3 and w_4 are conjugated in the quadratic extension $\mathbb{E}_4$ of $\mathbb{Q}_2$, with norm $w_3w_4=-\frac{1}{4}$ wich implies that $|w_3|=|w_4|=\left|-\frac{1}{4}\right|_2^{\frac{1}{2}}=2$. The period of w_3 and w_4 is 2.

More generally, one can prove that for any integer $\nu\geq 1$, if w is a ν-periodic point in $\mathbb{C}_2$ different from 1, then $|w|_2=2$.

$-\dagger\dagger-$ One immediately verifies that if the prime p is different from 2, then since the leading cœfficient of the Chebyshev polynomial T_n, $n\geq 2$, is equal to 2^{n-1}, then the leading cœfiicient of the reduced polynomial modulo p is the class of 2^{n-1} that is different from 0. Hence the polynomial $T_n, n\geq 2$, has good reduction modulo p and one deduces from a well known theorem of Morton and Silverman ([6]) that the *p-adic Julia set of T_n is the empty set.*
The congruences modulo 2 for the Chebyshev polynomials $T_n, n\geq 2$ quoted above show that these polynomials have bad reduction modulo 2. However, one can prove directly that the p-adic Julia set of any polynomial Chebyshev T_n of degree $n\geq 2$, is the *empty set, regardless the prime number p is 2 or odd.*

References

[1] Y. Amice, *Les nombres p-adiques*, PUF, Paris 1975.

[2] A. F. Beardon, *Iteration of rational functions*, Springer, GTM 132, New York, Berlin, Heidelberg, 2000.

[3] F. Q. Gouvea, *p-adic Numbers, An introduction*, Universitext, Springer, Berlin 1997

[4] A. Yu. Khrennikov, *Non-archimedan ananlysis* : *quantum paradoxes, dyamical systems and biological models*, Kluwer Academic Publishers, Dordrecht, 1997.

[5] A. Yu. Khrennikov and M. Nilson, *p-Adic deterministic and random dynamics*, Kluwer Academic Publishers, Dordrecht, 2004.

[6] P. Morton and J. Silverman, Periodic points, multiplicities and dynamical units, *J. Reine Angew. Math.* 461 (1995), no.2,183-208.

[7] A. M. Robert, Le théorème des accroissements finis p-adique, *Annales Mathématiques Blaise Pascal*, vol. 2, N°1, 1995.

[8] A. M. Robert, *A course in p-adic analysis*, Springer-Verlag, New York-Heidelberg-Berlin, 2000.

[9] M. Zuber, *Propriétés p–adiques des polynômes classiques,*
Thèse, Université de Neuchatel, (Suisse), 03 décembre 1992.

Laboratoire de Mathématiques, UMR 6620
CNRS - Université Blaise Pascal
Complexe Scientifique des Cézeaux
63 177 Aubière Cedex - France
e-mail : Bertin.Diarra@math.univ-bpclermont.fr