Radial Solutions of Non-Archimedean Pseudo-Differential Equations

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NOTATION.
Let K be a non-Archimedean local field, that is a non-discrete totally disconnected locally compact topological field, endowed with an absolute value $|\cdot|_K$.

q is the cardinality of the residue field.

Spaces of test functions and distributions:

$\mathcal{D}(K)$ is the set of all locally constant complex-valued functions on K with compact supports (with the topology of double inductive limit). The strong conjugate space $\mathcal{D}'(K)$ is called the space of Bruhat-Schwartz distributions.

$$\Psi(K) = \{\psi \in \mathcal{D}(K) : \ \psi(0) = 0\},$$

$$\Phi(K) = \left\{\varphi \in \mathcal{D}(K) : \ \int\limits_K \varphi(x)\,dx = 0\right\}.$$

The Fourier transform $\mathcal{F}$ is a linear isomorphism from $\Psi(K)$ onto $\Phi(K)$, thus also from $\Phi'(K)$ onto $\Psi'(K)$. The spaces $\Phi(K)$ and $\Phi'(K)$ are called the Lizorkin spaces (of the second kind) of test functions and distributions respectively.

Two distributions differing by a constant summand coincide as elements of $\Phi'(K)$.

Fractional differentiation operator D^α, $\alpha > 0$:

$$(D^\alpha \varphi)(x) = \mathcal{F}^{-1}\left[|\xi|^\alpha (\mathcal{F}(\varphi))(\xi)\right](x).$$

D^α does not act on the space $\mathcal{D}(K)$, since the function $\xi \mapsto |\xi|^\alpha$ is not locally constant. On the other hand, $D^\alpha : \ \Phi(K) \to \Phi(K)$ and $D^\alpha : \ \Phi'(K) \to \Phi'(K)$, and that was a motivation to introduce these spaces.

The operator D^α can also be represented as a hypersingular integral operator:

$$(D^\alpha \varphi)(x) = \frac{1 - q^\alpha}{1 - q^{-\alpha-1}} \int\limits_K |y|^{-\alpha-1}[\varphi(x-y) - \varphi(x)]\, dy.$$

This expression makes sense for wider classes of functions.

Definition of $D^{-\alpha}$, $\alpha > 0$:

$$\left(D^{-\alpha}\varphi\right)(x) = (f_\alpha * \varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_K |x-y|_K^{\alpha-1}\varphi(y)\,dy,\ \varphi \in \mathcal{D}(K),$$

$(\alpha \neq 1)$,and

$$\left(D^{-1}\varphi\right)(x) = \frac{1-q}{q\log q} \int\limits_K \log|x-y|_K \varphi(y)\,dy.$$

Then $D^{\alpha}D^{-\alpha} = I$ on $\mathcal{D}(K)$, if $\alpha \neq 1$. This property remains valid on $\Phi(K)$ also for $\alpha = 1$.

V. S. Vladimirov: *Properties of D^α are complicated.*

As an operator on $L_2(\mathbb{Q}_p)$, it has a point spectrum of infinite multiplicity.

The equation $D_t^\alpha u - D_x^\alpha u = 0$ has no fundamental solution.

However there is a well-developed theory of this equation and a more general one, with several spatial variables, with the assumption that a solution is radial in t, that is depends on $|t|_K$:

A. N. Kochubei, A non-Archimedean wave equation, *Pacif. J. Math.* **235** (2008), 245–261.

Lemma 1

If a function $u = u(|x|_K)$ *is such that*

$$\sum_{k=-\infty}^{m} q^k \left| u(q^k) \right| < \infty, \quad \sum_{l=m}^{\infty} q^{-\alpha l} \left| u(q^l) \right| < \infty,$$

for some $m \in \mathbb{Z}$*, then for each* $n \in \mathbb{Z}$ *the hypersingular integral expression for* $D^\alpha \varphi$ *with* $\varphi(x) = u(|x|_K)$ *exists for* $|x|_K = q^n$*, depends only on* $|x|_K$*, and*

$$(D^\alpha u)(q^n) = d_\alpha \left(1 - \frac{1}{q}\right) q^{-(\alpha+1)n} \sum_{k=-\infty}^{n-1} q^k u(q^k)$$
$$+ q^{-\alpha n - 1} \frac{q^\alpha + q - 2}{1 - q^{-\alpha - 1}} u(q^n) + d_\alpha \left(1 - \frac{1}{q}\right) \sum_{l=n+1}^{\infty} q^{-\alpha l} u(q^l).$$

Definition

We say that the action $D^\alpha u$, $\alpha > 0$, on a radial function u is defined in the strong sense, if the function u satisfies the conditions of Lemma 1, which gives the expression of $D^\alpha u(|x|_K)$, $|x|_K \neq 0$, and there exists the limit

$$D^\alpha u(0) \stackrel{def}{=} \lim_{x \to 0} D^\alpha u(|x|_K).$$

It is evident from the hypersingular integral formula that D^α annihilates constant functions (recall that in $\Phi'(K)$ they are equivalent to zero). Therefore $D^{-\alpha}$ is not the only possible choice of the right inverse to D^α. In particular, we will use

$$(I^\alpha \varphi)(x) = (D^{-\alpha}\varphi)(x) - (D^{-\alpha}\varphi)(0).$$

This is defined initially for $\varphi \in \mathcal{D}(K)$. It is seen from the ultrametric property of the absolute value that

$$(I^\alpha \varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_{|y|_K \le |x|_K} \left(|x-y|_K^{\alpha-1} - |y|_K^{\alpha-1}\right) \varphi(y)\, dy,\ \alpha \ne 1,$$

and

$$(I^1 \varphi)(x) = \frac{1-q}{q \log q} \int\limits_{|y|_K \le |x|_K} \left(\log|x-y|_K - \log|y|_K\right) \varphi(y)\, dy.$$

Let us calculate $I^\alpha u$ for a radial function $u = u(|x|_K)$. Obviously, $(I^\alpha u)(0) = 0$ whenever I^α is defined.

Lemma 2

Suppose that

$$\sum_{k=-\infty}^{m} \max\left(q^k, q^{\alpha k}\right)\left|u(q^k)\right| < \infty, \alpha \neq 1;$$

$$\sum_{k=-\infty}^{m} |k| q^k \left|u(q^k)\right| < \infty,\ \alpha = 1,$$

for some $m \in \mathbb{Z}$. Then $I^\alpha u$ exists, it is a radial function, and for any $x \neq 0$,

$$(I^\alpha u)(|x|_K) = q^{-\alpha}|x|_K^\alpha u(|x|_K) + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_{|y|_K<|x|_K} \left(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\right) u(|y|_K)\, dy, \quad \alpha \neq 1,$$

and

$$(I^1 u)(|x|_K) = q^{-1}|x|_K u(|x|_K) + \frac{1-q}{q\log q} \int\limits_{|y|_K<|x|_K} (\log |x|_K - \log |y|_K)\, u(|y|_K)\, dy.$$

Lemma 3

Suppose that for some $m \in \mathbb{Z}$,

$$\sum_{k=-\infty}^{m} \max\left(q^k, q^{\alpha k}\right)\left|v(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty}\left|v(q^l)\right| < \infty,$$

if $\alpha \neq 1$, *and*

$$\sum_{k=-\infty}^{m} |k| q^k \left|v(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} l \left|v(q^l)\right| < \infty,$$

if $\alpha = 1$. *Then there exists* $\left(D^\alpha I^\alpha v\right)\left(|x|_K\right) = v(|x|_K)$ *for any* $x \neq 0$.

Using Lemma 3, we can consider the simplest Cauchy problem

$$D^\alpha u(|x|_K) = f(|x|_K), \quad u(0) = 0,$$

where f is a continuous function, such that

$$\sum_{l=m}^{\infty} \left| f(q^l) \right| < \infty, \text{ if } \alpha \neq 1, \text{ or } \sum_{l=m}^{\infty} l \left| f(q^l) \right| < \infty, \text{ if } \alpha = 1.$$

The unique strong solution is $u = I^\alpha f$. Therefore on radial functions, the operators D^α and I^α behave like the fractional derivative and fractional integral of real analysis. An example of a different behavior in the non-Archimedean case:

Let $f(|x|_K) \equiv 1$, $x \in K$. *Then* $(I^\alpha f)(|x|_K) \equiv 0$.

In the class of radial functions $u = u(|x|_K)$, we consider the Cauchy problem

$$D^\alpha u + a(|x|_K)u = f(|x|_K), \quad x \in K, \tag{1}$$

$$u(0) = 0, \tag{2}$$

where a and f are continuous functions, that is they have finite limits $a(0)$ and $f(0)$, as $x \to 0$.
Looking for a solution of the form $u = I^\alpha v$, where v is a radial function, we obtain formally an integral equation

$$v(|x|_K) + a(|x|_K)\,(I^\alpha v)\,(|x|_K) = f(|x|_K), \quad x \in K.$$

By Lemma 2, the latter equation can be written in the form

$$\left[1+q^{-\alpha}a(|x|_K)|x|_K^{\alpha}\right]v(|x|_K)$$
$$+c_\alpha a(|x|_K)\int\limits_{|y|_K<|x|_K}\left(|x|_K^{\alpha-1}-|y|_K^{\alpha-1}\right)v(|y|_K)\,dy=f(|x|_K),\ x\neq 0,$$

where $c_\alpha=\dfrac{1-q^{-\alpha}}{1-q^{\alpha-1}}$.
Since a is continuous, there exists such $N\in\mathbb{Z}$ that

$$q^{-\alpha}a(|x|_K)|x|_K^{\alpha}<1\quad\text{for } |x|_K\leq q^N.$$

On the ball $B_N=\left\{x\in K:\ |x|_K\leq q^N\right\}$, the equation takes the form

$$v(|x|_K)+\int\limits_{|y|_K<|x|_K}k_\alpha(x,y)v(|y|_K)\,dy=F(|x|_K)\qquad(3)$$

where for $\alpha \neq 1$,

$$k_\alpha(x,y) = \left[1 + q^{-\alpha} a(|x|_K)|x|_K^\alpha\right]^{-1} c_\alpha a(|x|_K)\left(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\right)$$

$(x \neq 0)$, $k_\alpha(0,y) = 0$, $F(|x|_K) = [1 + q^{-\alpha} a(|x|_K)|x|_K^\alpha]^{-1} f(|x|_K)$.
If $\alpha = 1$, then

$$k_1(x,y) = \frac{1-q}{q\log q}\left[1 + q^{-1} a(|x|_K)|x|_K\right]^{-1} a(|x|_K)\left(\log|x|_K - \log|y|_K\right)$$

$$(x \neq 0),\ k_1(0,y) = 0,\ F(|x|_K) = \left[1 + q^{-1} a(|x|_K)|x|_K\right]^{-1} f(|x|_K).$$

If we construct a solution on B_N, and if

$$a(|x|_K) \neq -q^{\alpha m} \quad \text{for any } x \in K,\ m \in \mathbb{Z}, \tag{4}$$

we will be able to construct a solution **successively** for all $x \in K$. Further on, the condition (4) is satisfied.

Theorem 1

For each $\alpha > 0$, the integral equation (3) has a unique continuous solution on B_N.

The integral operator in (3) is compact on $C(B_N)$ and has no nonzero eigenvalues.

In general, the function $u = I^{\alpha} v$ satisfies (1) in the sense of distributions from Φ'. The initial condition (2) is satisfied automatically.

Let us find additional conditions on a and f, under which this construction gives a strong solution of the Cauchy problem (1)-(2). A strong solution is unique in the class of functions $u = I^{\alpha} v$ where v is a continuous radial function, such that $\sum\limits_{l=m}^{\infty} \left| v(q^l) \right| < \infty$ for sum $m \in \mathbb{Z}$.

Theorem 2

Suppose that

$$|a(|x|_K)| \le C|x|_K^{-\alpha-\varepsilon}, \quad |f(|x|_K)| \le C|x|_K^{-\varepsilon}, \quad \varepsilon > 0, C > 0,$$

as $|x|_K > 1$. *Then* $u = I^{\alpha} v$ *is a strong solution of the Cauchy problem (4.1)-(4.2).*

Instead of (2), one can consider an inhomogeneous initial condition $u(0) = u_0$, $u_0 \in \mathbb{C}$. Looking for a solution in the form $u = u_0 + I^\alpha v$, $v = v(|x|_K)$, we obtain the integral equation

$$v(|x|_K) + a(|x|_K)\,(I^\alpha v)\,(|x|_K) = f(|x|_K) - a(|x|_K)u_0,$$

which can be studied under the same assumptions.

All the above results carry over to the case of a matrix-valued coefficient $a(|x|_K)$ and vector-valued solutions. In this case, to obtain a strong solution, it is sufficient to demand that the spectrum of each matrix $a(|x|_K)$, $x \in K$, does not intersect the set $\{-q^N, N \in \mathbb{Z}\}$.