Minimal decomposition of linear fractional transformations on the projective line over $\mathbb{Q}_p$

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p-Adic Methods for Modeling of Complex Systems

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Outline

Introduction

I. The p-adic numbers

- $p \geq 2$ a prime number.

$$\forall n \in \mathbb{N},\ n = \sum_{i=0}^{N} a_i p^i \ (a_i = 0, 1, \cdots, p-1)$$

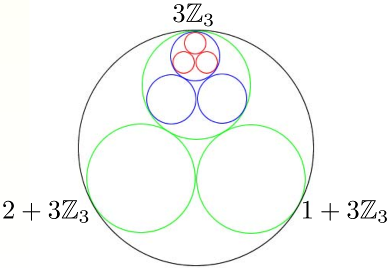
- Ring $\mathbb{Z}_p$ of p-adic integers :

$$\mathbb{Z}_p \ni x = \sum_{i=0}^{\infty} a_i p^i.$$

- Field $\mathbb{Q}_p$ of p-adic numbers : fraction field of $\mathbb{Z}_p$:

$$\mathbb{Q}_p \ni x = \sum_{i=v(x)}^{\infty} a_i p^i, \quad (\exists v(x) \in \mathbb{Z}).$$

Absolute value : $|x|_p = p^{-v(x)}$, metric : $d(x, y) = |x - y|_p$.



II. Arithmetic in $\mathbb{Q}_p$

Addition and multiplication : similar to the decimal way.
"Carrying" from left to right.

Example : $x = (p-1) + (p-1) \times p + (p-1) \times p^2 + \cdots$, then

- $x + 1 = 0$. So,

$$-1 = (p-1) + (p-1) \times p + (p-1) \times p^2 + \cdots .$$

- $2x = (p-2) + (p-1) \times p + (p-1) \times p^2 + \cdots .$

We also have substraction and division.

Then we can define polynomials and rational maps.

III. Equicontinuous dynamics

A dynamical system is a couple (X, T) where $T : X \to X$ is a transformation on the space X.
Example : $(\mathbb{Z}_p, f)$ with $f \in \mathbb{Z}_p[x]$ being a polynomial.

We say $T : X \to X$ is equicontinuous if

$$\forall \epsilon > 0, \exists \delta > 0 \ \text{ s. t. } \ d(T^n x, T^n y) < \epsilon \ (\forall n \geq 1, \forall d(x, y) < \delta).$$

Theorem

Let X be a compact metric space and $T : X \to X$ be an *equicontinuous transformation*. Then the following statements are equivalent :
(1) T is **minimal** (every orbit is dense).
(2) T is **uniquely ergodic** (there is a unique invariant measure).
(3) T is **ergodic** for any/some invariant measure with X as its support.

- Fact : 1-Lipschitz transformation is equicontinuous.
- Fact : Polynomial $f \in \mathbb{Z}_p[x] : \mathbb{Z}_p \to \mathbb{Z}_p$ is equicontinuous.

IV. Study on p-adic dynamical dystems

- Oselies-Zieschang 1975 : automorphisms of $\mathbb{Z}_p$
  Herman-Yoccoz 1983 : complex p-adic dynamical systems
  Volovich 1987 : p-adic string theory
- Lubin 1994 ; Zelenov 2013 : p-adic analytic transformations.
- Thiran-Verstegen-Weyers 1989 ; Woodcock-Smart 1998 ; Fan-Liao-Wang-Zhou 2007 ; Benedetto-Briend-Perdry 2007 ; Kingsbery-Levin-Preygel-Silva 2009 : Chaotic p-adic (polynomial) dynamical systems.
- Anashin 1994 : 1-Lipschitz transformation (Mahler series)
  Yurova 2010, 2012 ; Anashin-Khrennikov-Yurova 2011, 2012 ; Lin-Shi-Yang 2012 ; Jeong 2012 ; Khrennikov-Yurova 2013 : 1-Lipschitz transformation (Van der Put series)
- Coelho-Parry 2001 : ax and distribution of Fibonacci numbers
  Gundlach-Khrennikov-Lindahl 2001 : x^n
  Diarra-Sylla 2013 : Chebyshev polynomials
- $\cdots\cdots$ (see Vivaldi's database)

Affine polynomial dynamical systems on $\mathbb{Q}_p$

I. Polynomial dynamical systems on $\mathbb{Z}_p$

- Let $f \in \mathbb{Z}_p[x]$ be a polynomial with coefficients in $\mathbb{Z}_p$.
- Polynomial dynamical systems : $f : \mathbb{Z}_p \to \mathbb{Z}_p$, noted as $(\mathbb{Z}_p, f)$.

Theorem (Ai-Hua Fan, L 2011) minimal decomposition

Let $f \in \mathbb{Z}_p[x]$ with $\deg f \geq 2$. The space $\mathbb{Z}_p$ can be decomposed into three parts :

$$\mathbb{Z}_p = A \sqcup B \sqcup C,$$

where

- A is the finite set consisting of all periodic orbits ;
- $B := \sqcup_{i \in I} B_i$ (I finite or countable)
  $\to$ B_i : finite union of balls,
  $\to$ $f : B_i \to B_i$ is minimal ;
- C is attracted into $A \sqcup B$.

II. Conjugate classes

Given a positive integer sequence $(p_s)_{s\geq 0}$ such that $p_s | p_{s+1}$.

Profinite groupe : $\mathbb{Z}_{(p_s)} := \lim\limits_{\leftarrow} \mathbb{Z}/p_s\mathbb{Z}$.

Odometer : The transformation $\tau : x \mapsto x+1$ on $\mathbb{Z}_{(p_s)}$.

Theorem (J.-L. Chabert, A.-H. Fan, Y. Fares 2009)

Let E be a compact set in $\mathbb{Z}_p$ and $f : E \to E$ a 1-lipschitzian transformation. If the dynamical system (E, f) is minimal, then

- (E, f) is conjugate to the odometer $(\mathbb{Z}_{(p_s)}, \tau)$ where (p_s) is determined by the structure of E.

Theorem (Fan-L 2011 : Minimal components of polynomials)

Let $f \in \mathbb{Z}_p[X]$ be a polynomial and $O \subset \mathbb{Z}_p$ a clopen set, $f(O) \subset O$. Suppose $f : O \to O$ is minimal.

- If $p \geq 3$, then $(O, f|_O)$ is conjugate to the odometer $(\mathbb{Z}_{(p_s)}, \tau)$ where
$$(p_s)_{s\geq 0} = (k, kd, kdp, kdp^2, \dots) \quad (1 \leq k \leq p, d|(p-1)).$$
- If $p = 2$, then $(O, f|_O)$ is conjugate to $(\mathbb{Z}_2, x+1)$.

III. Affine polynomials on $\mathbb{Z}_p$

Let $T_{a,b}x = ax + b \quad (a, b \in \mathbb{Z}_p)$. Denote

$$\mathbb{U} = \{z \in \mathbb{Z}_p : |z| = 1\}, \quad \mathbb{V} = \{z \in \mathbb{U} : \exists m \geq 1, \text{s.t. } z^m = 1\}.$$

Easy cases :

1. $a \in \mathbb{Z}_p \setminus \mathbb{U} \quad \Rightarrow$ one attracting fixed point $b/(1-a)$.
2. $a = 1, b = 0 \ \Rightarrow$ every point is fixed.
3. $a \in \mathbb{V} \setminus \{1\} \ \Rightarrow$ every point is on a ℓ-periodic orbit, with ℓ the smallest integer $\geqslant 1$ such that $a^\ell = 1$.

Theorem (AH. Fan, MT. Li, JY. Yao, D. Zhou 2007) Case $p \geq 3$:

4. $a \in (\mathbb{U} \setminus \mathbb{V}) \cup \{1\},\ v_p(b) < v_p(1-a) \ \Rightarrow p^{v_p(b)}$ minimal parts.
5. $a \in \mathbb{U} \setminus \mathbb{V},\ v_p(b) \geq v_p(1-a) \Rightarrow (\mathbb{Z}_p, T_{a,b})$ is conjugate to $(\mathbb{Z}_p, ax)$.

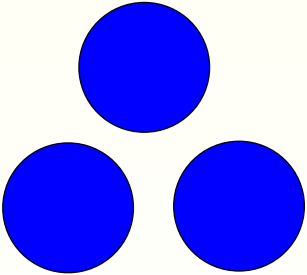
Decomposition : $\mathbb{Z}_p = \{0\} \sqcup \sqcup_{n \geq 1} p^n \mathbb{U}$.

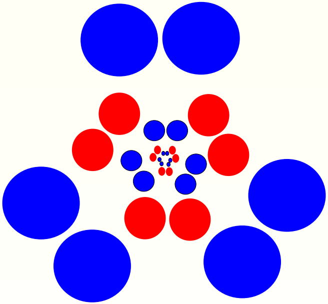
(1) One fixed point $\{0\}$.

(2) All $(p^n\mathbb{U}, ax)(n \geq 0)$ are conjugate to $(\mathbb{U}, ax)$.

For $(\mathbb{U}, T_{a,0})$: $p^{v_p(a^\ell - 1)}(p-1)/\ell$ minimal parts, with ℓ the smallest integer $\geqslant 1$ such that $a^\ell \equiv 1 (\mathrm{mod}\ p)$.

Two typical decompositions of $\mathbb{Z}_p$





Theorem (Fan-Li-Yao-Zhou 2007) Case $p = 2$:

4. $a \in (\mathbb{U} \setminus \mathbb{V}) \cup \{1\},\ v_p(b) < v_p(1-a)$.
   - $v_p(b) = 0 \Rightarrow p^{v_p(a+1)-1}$ minimal parts.
   - $v_p(b) > 0 \Rightarrow p^{v_p(b)}$ minimal parts.
5. $a \in \mathbb{U} \setminus \mathbb{V},\ v_p(b) \geq v_p(1-a)$

   $\Rightarrow (\mathbb{Z}_p, T_{a,b})$ is conjugate to $(\mathbb{Z}_p, ax)$.

   Decomposition : $\mathbb{Z}_p = \{0\} \sqcup \sqcup_{n \geq 1} p^n \mathbb{U}$.

   (1) One fixed point $\{0\}$.

   (2) All $(p^n \mathbb{U}, ax)(n \geq 0)$ are conjugate to $(\mathbb{U}, ax)$.

   For $(\mathbb{U}, T_{a,0})$: $2^{v_2(a^2-1)-2}$ minimal parts.

Remark : For the case $p = 2$, all minimal parts (except for the periodic orbits) are conjugate to $(\mathbb{Z}_2, x+1)$.

IV. Affine polynomials on $\mathbb{Q}_p$

Let φ be an affine map defined by

$$\varphi(x) = ax + b \ (a, b \in \mathbb{Q}_p, a \neq 0, (a, b) \neq (1, 0)).$$

If $|a| \neq 1$: easy ! For $|a| = 1$, we have the following **conjugacy** :

- $a \neq 1$:

$$\begin{array}{ccc} \mathbb{Q}_p & \xrightarrow{\ ax+b\ } & \mathbb{Q}_p \\ {\scriptstyle x - \frac{b}{1-a}} \downarrow & & \downarrow {\scriptstyle x - \frac{b}{1-a}} \\ \mathbb{Q}_p & \xrightarrow{\ \color{red}{ax}\ } & \mathbb{Q}_p \end{array}$$

- $a = 1$:

$$\begin{array}{ccc} \mathbb{Q}_p & \xrightarrow{\ x+b\ } & \mathbb{Q}_p \\ {\scriptstyle \frac{x}{b}} \downarrow & & \downarrow {\scriptstyle \frac{x}{b}} \\ \mathbb{Q}_p & \xrightarrow{\ \color{red}{x+1}\ } & \mathbb{Q}_p \end{array}$$

V. Affine polynomials on $\mathbb{Q}_p$-continued

Theorem (AH. Fan, Y. Fares 2011)

If $K = \mathbb{Q}_p$, then

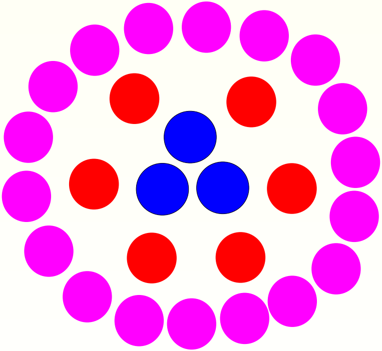
1. $\varphi(x) = x + 1 : \mathbb{Q}_p = \mathbb{Z}_p \cup \bigcup_{n=1}^{\infty} p^n \mathbb{U}$.
   - $\mathbb{Z}_p$ is minimal.
   - $p^n\mathbb{U}$ contains $p^{n-1}(p-1)$ minimal balls with radius 1.
2. $\varphi(x) = ax$ (a is not a root of unity) : $\mathbb{Q}_p = \{0\} \cup \bigcup_{n\in\mathbb{Z}} p^n\mathbb{U}$.
   - 0 is fixed.
   - All subsystems on $p^n\mathbb{U}$ are conjugate to $(\mathbb{U}, \varphi)$.

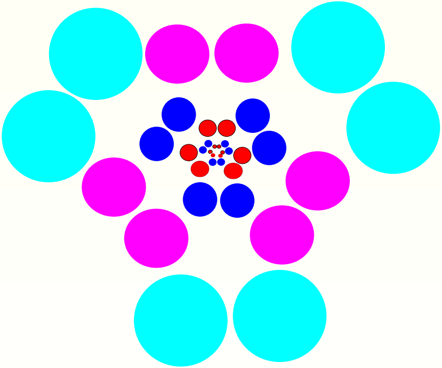
     For $(\mathbb{U}, \varphi)$:

     (1) Case $p \geq 3$: $p^{v_p(a^\ell - 1)}(p-1)/\ell$ minimal balls of same radius, with ℓ the smallest integer $\geqslant 1$ such that $a^\ell \equiv 1 (\mathrm{mod}\ p)$.

     (2) Case $p = 2$: $2^{v_2(a^2-1)-2}$ minimal balls of same radius.

Two typical decompositions of $\mathbb{Q}_p$





P-adic linear frctional transformations

I. Projective line over $\mathbb{Q}_p$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}_p^2 \setminus \{(0,0)\}$, we say that $(x_1, y_1) \sim (x_2, y_2)$ if $\exists \lambda \in \mathbb{Q}_p^*$ s.t.

$$x_1 = \lambda x_2 \text{ and } y_1 = \lambda y_2.$$

Projective line over $\mathbb{Q}_p$:

$$\mathbb{P}^1(\mathbb{Q}_p) := (\mathbb{Q}_p^2 \setminus \{(0,0)\}) / \sim$$

Spherical metric : Let $P = [x_1, y_1], Q = [x_2, y_2] \in \mathbb{P}^1(\mathbb{Q}_p)$, define

$$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$$

Viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $K \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \qquad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \le 1\,; \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$

II. Linear fractional transformations

Let

$$\phi(x) = \frac{ax+b}{cx+d} \quad \text{with } a,b,c,d \in \mathbb{Q}_p,\ ad-bc \neq 0,$$

which induces an 1-to-1 map $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$.

- $\phi(-d/c) = \infty$, and $\phi(\infty) = a/c$.
- ϕ is a composition of $\phi_1(x) = \alpha x, \phi_2(x) = x + \beta, \phi_3(x) = x \mapsto 1/x$.
- if $\mathbb{D}(a,r)$ is a disk in $\mathbb{P}^1(\mathbb{Q}_p)$, then $\phi(\mathbb{D}(a,r))$ is also a disk. (a disk in $\mathbb{P}^1(\mathbb{Q}_p)$ is a disk in $\mathbb{Q}_p$ or a complement of a disk in $\mathbb{Q}_p$.)

1. $\phi_1(\mathbb{D}(a,r)) = \mathbb{D}(\alpha a, r|\alpha|_p)$.
2. $\phi_2(\mathbb{D}(a,r)) = \mathbb{D}(a+\beta, r)$.
3. if $0 \in \mathbb{D}(a,r)$, $\phi_3(\mathbb{D}(a,r)) = \mathbb{P}^1(K) \setminus \overline{\mathbb{D}}(0, 1/r)$.
4. if $0 \notin \mathbb{D}(a,r)$, $\phi_3(\mathbb{D}(a,r)) = \mathbb{D}(a^{-1}, r|a|_p^{-2})$.

III. Some studies on linear fractional transformations

- Diao-Silva 2011 : There is no minimal rational maps on $\mathbb{Q}_p$.

  (Comments : They did not include the infinity point. We need study the rational maps on $\mathbb{P}^1(\mathbb{Q}_p)$.)

- Dragovich-Khrennikov-Mihajlovic 2007 : Linear fractional transformations on adelic space.

IV. Fixed points and dynamics

The dynamics of ϕ depends on its fixed points which are the solution of

$$\frac{ax+b}{cx+d} = x \Leftrightarrow cx^2 + (d-a)x - b = 0.$$

Discriminant : $\Delta = (d-a)^2 + 4bc$.

- If $\Delta = 0$, then ϕ has only **one fixed point** x_0 in $\mathbb{Q}_p$ and $\phi(x)$ is conjugate to a translation $\psi(x) = x + \alpha$ for some $\alpha \in \mathbb{Q}_p$ by $g(x) = \frac{1}{x-x_0}$.
- If $\Delta \neq 0$ and $\sqrt{\Delta} \in \mathbb{Q}_p$, then ϕ has **two fixed points** $x_1, x_2 \in \mathbb{Q}_p$ and ϕ is conjugate to a multiplication $x \mapsto \beta x$ for some $\beta \in \mathbb{Q}_p$ by $g(x) = \frac{x-x_2}{x-x_1}$.
- If $\Delta \neq 0$ and $\sqrt{\Delta} \notin \mathbb{Q}_p$, then ϕ has **no fixed point** in $\mathbb{Q}_p$. But ϕ has two fixed points $x_1, x_2 \in \mathbb{Q}_p(\sqrt{\Delta})$. So we will study the dynamics of ϕ on $\mathbb{P}^1(\mathbb{Q}_p(\sqrt{\Delta}))$ then its restriction on $\mathbb{P}^1(\mathbb{Q}_p)$.

V. Notations

- K is a finite extension of $\mathbb{Q}_p$.
- Still denote by $|\cdot|_p$ the extended absolute value of K.
- Degree : $d = [K : \mathbb{Q}_p]$. Ramification index : e
- Valuation function : $v_p(x) := -\log_p(|x|_p)$. $\mathrm{Im}(v_p) = \frac{1}{e}\mathbb{Z}$.
- $\mathcal{O}_K := \{x \in K : |x|_p \leq 1\}$: the local ring of K,
  $\mathcal{P}_K := \{x \in K : |x|_p < 1\}$: its maximal ideal.
- Residual field : $\mathbb{K} = \mathcal{O}_K/\mathcal{P}_K$. Then $\mathbb{K} = \mathbb{F}_{p^f}$, with $f = d/e$.

Quadratic extensions :

- 7 quadratic extensions of $\mathbb{Q}_2$:

$$\mathbb{Q}_2(\sqrt{-1}),\ \mathbb{Q}_2(\sqrt{\pm 2}),\ \mathbb{Q}_2(\sqrt{\pm 3}),\ \mathbb{Q}_2(\sqrt{\pm 6}).$$

- 3 quadratic extensions of $\mathbb{Q}_p (p \geq 3)$:

$$\mathbb{Q}_p(\sqrt{p}),\ \mathbb{Q}_p(\sqrt{N_p}),\ \mathbb{Q}_p(\sqrt{pN_p}),$$

  where N_p is the smallest quadratic non-residue module p.

VI. Uniformizer and representation

An element $\pi \in K$ is a uniformizer if $v_p(\pi) = 1/e$.

Define $v_\pi(x) := e \cdot v_p(x)$ for $x \in K$. Then $\mathrm{Im}(v_\pi) = \mathbb{Z}$, and $v_\pi(\pi) = 1$.

Let $C = \{c_0, c_1, \ldots, c_{p^f-1}\}$ be a fixed complete set of representatives of the cosets of $\mathcal{P}_K$ in $\mathcal{O}_K$. Then every $x \in K$ has a unique π-adic expansion of the form

$$x = \sum_{i=i_0}^{\infty} a_i \pi^i,$$

where $i_0 \in \mathbb{Z}$ and $a_i \in C$ for all $i \geq i_0$.

Example : For $\mathbb{Q}_p(\sqrt{p})$ $(p \geq 3)$, take $\pi = \sqrt{p}$, and

$$x = a_0 + a_1\sqrt{p} + a_2 p + a_3 p^{3/2} + a_4 p^2 + \cdots .$$

VII. Minimal decomposition (ϕ admits no fixed point)

Theorem (AH. Fan, SL. Fan, L, YF. Wang (preprint))

Suppose that ϕ *has no fixed points in* $\mathbb{P}^1(\mathbb{Q}_p)$ *and* $\phi^n \neq id$ *for each integer* $n > 0$. *Then*

1. *the system* $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ *is decomposed as a finite number of minimal subsystems ;*
2. *these minimal subsystems are topologically conjugate to each other ;*
3. *the number of minimal subsystems is determined by the number*

$$\lambda := \frac{(a+d)+\sqrt{\Delta}}{(a+d)-\sqrt{\Delta}}.$$

Denote

- $K = \mathbb{Q}_p(\sqrt{\Delta})$ be the quadratic extension of $\mathbb{Q}_p$ generated by $\sqrt{\Delta}$.
- π be an uniformizer of K
- $\mathbb{K}$ be the residue field of K.
- ℓ be the order in the group $\mathbb{K}^*$ of λ.

VIII. The case $p \geq 3$

Theorem (Fan-Fan-L-Wang, $K = \mathbb{Q}_p(\sqrt{N_p})$ is unramified)

The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as $((p+1)p^{v_p(\lambda^\ell - 1)-1})/\ell$ minimal subsystems. Each subsystem is topologically conjugate to the adding machine on an odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (\ell, \ell p, \ell p^2, \cdots)$.

Theorem (Fan-Fan-L-Wang, $K = \mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{pN_p})$ is ramified)

(1) *If $|a+d|_p > |\sqrt{\Delta}|_p$, then $\lambda = 1 \pmod{\pi}$. The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as $2p^{(v_\pi(\lambda^p - 1)-3)/2}$ minimal subsystems. Moreover, each minimal subsystem is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (1, p, p^2, \cdots)$.*

(2) *If $|a+d|_p < |\sqrt{\Delta}|_p$, then $\lambda = -1 \pmod{\pi}$. The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as $p^{(v_\pi(\lambda^p + 1)-3)/2}$ minimal subsystems. Moreover, each minimal subsystem is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (2, 2p, 2p^2, \cdots)$.*

IX. The case $p = 2$

Theorem (FFLW, $K = \mathbb{Q}_2(\sqrt{-3})$ is unramified)

The dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ *is decomposed as* $3 \cdot 2^{v_2(\lambda^{2\ell}-1)-2}/\ell$ *minimal subsystems. Moreover, each minimal system is conjugate to the adding machine on the odometer* $\mathbb{Z}_{(p_s)}$ *with* $(p_s) = (\ell, \ell 2, \ell 2^2, \cdots)$.

Theorem (FFLW, $K = \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-6}), \mathbb{Q}_2(\sqrt{6})$ ramified)

(1) *If* $|a+d|_2 > |\sqrt{\Delta}|_2$, *then* $v_\pi(\lambda - 1) \geq 3$ *is odd and the dynamical system* $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ *is decomposed as* $2^{(v_\pi(\lambda-1)-1)/2}$ *minimal subsystems.*

(2) *If* $|a+d|_2 < |\sqrt{\Delta}|_2$, *then* $v_\pi(\lambda + 1) \geq 3$ *is odd and the dynamical system* $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ *is decomposed as* $2^{(v_\pi(\lambda+1)-1)/2}$ *minimal subsystems. Moreover, each minimal system is conjugate to the adding machine on the odometer* $\mathbb{Z}_{(p_s)}$ *with* $(p_s) = (1, 2, 2^2, \cdots)$.

X. The case $p = 2$ (continued)

Theorem (FFLW, $K = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{3})$ is ramified)

(1) *If* $|a+d|_2 = |\sqrt{\Delta}|_2$, $v_\pi(\lambda^2+1) \geq 2$ *is even and the system* $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ *is decomposed as* $2^{(v_\pi(\lambda^2+1)-2)/2}$ *minimal subsystems.*

(2) *If* $|a+d|_2 > |\sqrt{\Delta}|_2$, *then* $v_\pi(\lambda-1) \geq 4$ *is even and the system* $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ *is decomposed as* $2^{v_\pi(\lambda-1)/2}$ *minimal subsystems.*

(3) *If* $|a+d|_2 < |\sqrt{\Delta}|_2$, $v_\pi(\lambda+1) \geq 4$ *is even and the system* $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ *is decomposed as* $2^{v_\pi(\lambda+1)/2}$ *minimal subsystems.*
Moreover, each minimal system is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ *with* $(p_s) = (1, 2, 2^2, \cdots)$.

XI. Minimal (ergodic) conditions

Corollary (FFLW, case $p \geq 3$)

The system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if one of the following conditions satisfied

(1) $K = \mathbb{Q}_p(\sqrt{\Delta})$ *is unramified,* $\ell = p + 1$ *and* $v_p(\lambda^\ell - 1) = 1$,

(2) $K = \mathbb{Q}_p(\sqrt{\Delta})$ *is ramified and* $v_\pi(\lambda^p + 1) = 3$.

Corollary (FFLW, case $p = 2$)

The system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is minimal if and only if one of the following conditions satisfied

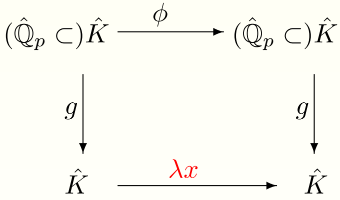
(1) $K = \mathbb{Q}_2(\sqrt{\Delta}) = \mathbb{Q}_2(\sqrt{-3})$, $\ell = 3$ *and* $v_2(\lambda^{2\ell} - 1) = 2$,

(2) $K = \mathbb{Q}_2(\sqrt{\Delta}) = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{3})$, $|a + b|_2 = |\sqrt{\Delta}|_2$ *and* $v_\pi(\lambda^2 + 1) = 2$.

Ideas and methods

I. Conjugacy and restriction

Let x_1, x_2 be the two fixed points in $K \setminus \mathbb{Q}_p$. Let $g(x) = \frac{x-x_2}{x-x_1}$.
Denote $\hat{K} = \mathbb{P}^1(K)$. Remark that $\hat{\mathbb{Q}}_p = \mathbb{P}^1(\mathbb{Q}_p)$ is invariant under ϕ.



Step I : Do minimal decomposition of $(K, \lambda x)$.
Step II : Find $g(\hat{\mathbb{Q}}_p)$ and determine the restriction $(g(\hat{\mathbb{Q}}_p), \lambda x)$.
Step III : Go back to $\hat{\mathbb{Q}}_p$.

II. Methods for minimal decomposition of $\mathbb{Z}_p, \mathbb{Q}_p, K$.

Fan, Li, Yao, Zhou : Fourier analysis.

Our methodes :

Theorem (Anashin 1994, Chabert, Fan and Fares 2009)

Let $X \subset \mathcal{O}_K$ be a compact set.
$\varphi : X \to X$ is minimal $\Leftrightarrow$
$\varphi_k : X/\pi^k\mathcal{O}_K \to X/\pi^k\mathcal{O}_K$ is minimal for all $k \geq 1$.

Predicting the behavior of φ_{k+1} by the structure of φ_k.
$\rightarrow$ Idea of Desjardins-Zieve 1994 (arXiv) and Zieve's Ph.D. Thesis 1996.

- Consider the cycle $(x_1, \ldots, x_k)$ in $\mathcal{O}_K/\pi^n\mathcal{O}_K$,
- Each x_i is lift to be p^f points $\{x_i + t\pi^n : 0 \leq t < p^f\}$ in $\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K$.

Linearization : $g := \varphi^k$,

$$g(x_1 + t\pi^n) \equiv x_1 + (a_n t + b_n)\pi^n \pmod{\pi^{n+1}}$$

with $a_n = g'(x_1), \quad b_n = \frac{g(x_1)-x_1}{\pi^n}$.

Linear maps Φ : $\Phi(t) = a_n t + b_n$.

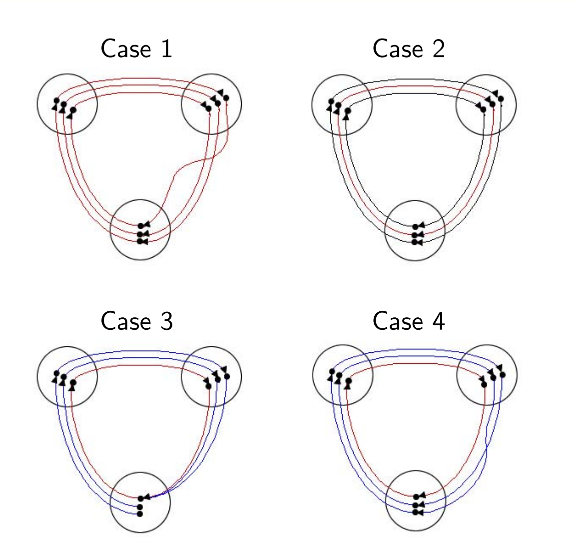
III. Ideas and methods (continued)

Lifts of the cycle $(x_1, \ldots, x_k)$:

Let $X_{n+1} = \{x_i + t\pi^n : 0 \leq t < p^f\}$

- $a_n \equiv 1, b_n \not\equiv 0 \mod \pi$: $\varphi_{n+1}|_{X_{n+1}}$ has p^{f-1} cycles of length pk. We say σ grows.
- $a_n \equiv 1, b_n \equiv 0 \mod \pi$: $\varphi_{n+1}|_{X_{n+1}}$ has p^f cycles of length k. We say σ splits.
- $a_n \equiv 0 \mod \pi$: $\varphi_{n+1}|_{X_{n+1}}$ has a single cycle of length k and the remaining points of X are mapped into this cycle by φ^k. We say σ grows tails.
- $a_n \not\equiv 0, 1 \mod \pi$: $\varphi_{n+1}|_{X_{n+1}}$ has a single cycle of length k and $(p^f - 1)/\ell$ cycles of length $k\ell$. We say σ partially splits.

Behavior of φ_{n+1}



IV. Subsystems and types

Let $\vec{E} = (E_1, E_2, \cdots)$ be a vector with $E_i \in \mathbb{N}^*$. A compact

$$\mathbb{X} = \bigsqcup_{i=1}^{k} (x_i + \pi^n \mathcal{O}_K)$$

is called of type $(k, \vec{E})$ if :

- It is a k-cycle growing at level n and all the lifts of this k-cycle split $E_1 - 1$ times.
- Then, all E_1-th generations of descendants grow and then all the lifts split $E_2 - 1$ times.
- Further, all the lifts of these descendants at level $E_1 + E_2$ split $E_3 - 1$ times, ...

If $\mathbb{X}$ is of type $(k, \vec{E})$, then $(\mathbb{X}, f)$ is decomposed into

- countable if the extension degree $d = 1$,
- uncountable (cardinality of $\mathbb{R}$) many, if $d > 1$,

minimal subsystems, each is conjugate to the odometer $(\mathbb{Z}_{(p_s)}, \tau)$ with

$$(p_s) = (k, \underbrace{kp, \cdots, kp}_{E_1}, \underbrace{kp^2, \cdots, kp^2}_{E_2}, \underbrace{kp^3, \cdots, kp^3}_{E_3}, \cdots).$$

If $\vec{E} = (e, e, e, \ldots)$, we call simply that X is of type (k, e).

V. Minimal decomposition for $\alpha x + \beta$ on K

We need only to treat $\varphi(x) = x + 1$ and $\varphi(x) = \alpha x$.
Let $\mathbb{U} = \{x \in K : |x|_p = 1\}$.

Theorem (Shi-Lei Fan and L, preprint)

1. $\varphi(x) = x + 1 : K = \mathcal{O}_K \cup \bigcup_{n=1}^{\infty} \pi^n \mathbb{U}$.
   - $\mathcal{O}_K$ is of type $(1, e)$.
   - $\pi^n \mathbb{U}$ contains $p^{(n-1)f}(p^f - 1)$ balls with radius 1, each is of type $(1, e)$.
2. $\varphi(x) = \alpha x$ (α is not a root of unity) : $K = \{0\} \cup \bigcup_{n \in \mathbb{Z}} \pi^n \mathbb{U}$.
   - 0 is fixed and all subsystems on $\pi^n \mathbb{U}$ are conjugate to $(\mathbb{U}, \varphi)$. For $(\mathbb{U}, \varphi)$: (1) Case $p \geq 3$: Denote by ℓ the smallest integer $\geqslant 1$ such that $\alpha^\ell \equiv 1 \pmod{\pi}$. The subsystem $\mathbb{U}$ is decomposed into

$$(p^f - 1) \cdot p^{v_\pi(\alpha^\ell - 1)f - f} / \ell$$

balls of same radius and each is of type $(\ell, \vec{E})$ where
$\vec{E} = (v_\pi(\frac{\alpha^{\ell p} - 1}{\alpha^\ell - 1}),\ v_\pi(\frac{\alpha^{\ell p^2} - 1}{\alpha^{\ell p} - 1}),\ \cdots,\ v_\pi(\frac{\alpha^{\ell p^{N+1}} - 1}{\alpha^{\ell p^N} - 1}),\ e,\ e,\ \cdots)$,
N is the largest integer such that $v_\pi((\alpha^{\ell p^{N+1}} - 1)/(\alpha^{\ell p^N} - 1)) \neq e$.

Theorem (Shi-Lei Fan and L, preprint)

For $(\mathbb{U}, \varphi)$:
(2) Case $p = 2$:
Denote by ℓ the smallest integer $\geqslant 1$ such that $\alpha^\ell \equiv 1 \pmod{\pi}$.
The subsystem $\mathbb{U}$ is decomposed into

$$(p^f - 1) \cdot p^{v_\pi(\alpha^\ell - 1)f - f} / \ell$$

compact sets and each compact set is of type $(\ell, \vec{E})$ with

$$\vec{E} = \Big(v_\pi(\alpha^\ell + 1),\ v_\pi(\alpha^{\ell p} + 1),\ \cdots,\ v_\pi(\alpha^{\ell p^N} + 1),\ e,\ e,\ \cdots\Big),$$

where N the biggest integer such that $v_\pi(\alpha^{\ell p^N} + 1) \neq e$.