Renormalization theory and renormalization group in the p-adic mirror

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$$
\begin{array}{ccc}
(R^d, R) & \longrightarrow & (R^d, \Gamma) \\
\downarrow & & \downarrow \\
(Q_p^d, R) & \longrightarrow & (Q_p^d, \Gamma)
\end{array}
$$

R^d — Euclidean space, Q_p^d— p—adic space, d— space dimension, Γ—Grassmann algebra.

As an example of bosonic theory we consider φ^4-theory with the action

$$H(\varphi; r, g) = H_0(\varphi; \alpha) + \int L(\varphi(x); r, g)\, dx, \qquad (1)$$

where

$$L(\varphi(x); r, g) = r\varphi^2(x) + g\varphi^4(x).$$

The Gaussian part is

$$H_0(\varphi;\alpha) = \frac{c(\alpha)}{2} \int |x-y|^{-\alpha}\varphi(x)\varphi(y)\,dxdy, \qquad (2)$$

while dx is the Lebesgue measure in the Euclidean case and Haar measure in p-adic case.

As an example of the fermionic thery we consider the four–component field $\psi^*(x) = (\overline{\psi}_1(x), \psi_1(x), \overline{\psi}_2(x), \psi_2(x))$ over Q_p^d, whose components are generators of tne Grassmann algebra. Let the Gibbs state describing this field be determined by the Hamiltonian

$$H(\psi^*;\alpha;r,g) = H_0(\psi^*;\alpha) + \int L(\psi^*(x);r,g)\,dx, \qquad (3)$$

$$H_0(\psi^*;\alpha) =$$

$$c(\alpha) \int ||x-y||^{-\alpha} \left(\overline{\psi}_1(x)\psi_1(y) + \overline{\psi}_2(x)\psi_2(y)\right)\,dxdy.$$

The Lagrangian is

$$L(\psi^*(x);r,g) = r(\overline{\psi}_1(x)\psi_1(x) + \overline{\psi}_2(x)\psi_2(x)) +$$

$$+g(\overline{\psi}_1(x)\psi_1(x)\overline{\psi}_2(x)\psi_2(x)).$$

The Gaussian Hamiltonian is invariant w.r.t.the group of scaling transformations $(S_\lambda(\alpha)\varphi)(x) = |\lambda|^{d-\alpha}\varphi(\lambda x)$, where $\alpha \in R$ is the parameter of this group.

The value $\alpha = d+2$ corresponds to the Laplace operator in the Euclidean case or its p-adic analog.

In the space of the Hamiltonians the transformation $S_\lambda(\alpha)$ acts by merely scaling the coupling constants: $S_\lambda(\alpha)(r, g) = (|\lambda|^{\alpha-d}r), |\lambda|^{2\alpha-3d}g)$

The discretization of the field φ is the field ξ over Z^d in the Euclidean case or T_p^d in the p-adic case such that

$$\xi(j) = \int \varphi(j + x)\chi(x)\, dx, \quad j \in T_p^d.$$

Here T_p^d is a lattice of p-adic fractional vectors (hierarchical lattice), $\chi(x)$ - characteristic function of the ball $Z_p^d = \{x \in Q_p^d : ||x||_p \leq 1\}$.

The discretization ζ of the field $(S_{p^{-1}}(\alpha)\varphi)(x)$ is hierarchical block-spin renormalization group transformation over the field ξ,

$$\zeta(j) = (r(\alpha)\xi)(j) = p^{-\alpha/2} \sum_{i\in B(j)} \xi(i),$$

where $B(j) = \{i \in T_p^d : ||i - p^{-1}j|| \leq p\}$ are the elementary blocks in the hierarchical lattice.

Particularly, discretization of the Gaussian field fermionic field ψ^* with Hamiltonian

$$H(\psi^*; \alpha; u, v) = H_0(\psi^*; \alpha) + \int L(\psi^*(x); u, v)\, dx,$$

is a a discrete fermionic field $\xi^*(j)$ with the Hamiltonian

$$H'(\xi^*; \alpha; r, g) = H'_0(\xi; \alpha) + \sum_{j\in T_p^d} L(\xi^*(j); r, g),$$

with the Gaussian part

$$H_0^{'}(\xi^*; \alpha) = \sum_{i,j} h(i, j; \alpha)\ (\,\overline{\xi}_1(i)\xi_1(j) + \overline{\xi}_2(i)\xi_2(j)),$$

$$h(i, j; \alpha) = c_1(\alpha)\,(1 - \delta_{i,j})\,||i - j||^{-\alpha} + c_1(\alpha)\delta_{i,j}.$$

Thus, the discretization of the continuum field leads to the hierarchical model with the same potential $L(\xi^*; r, g)$, where the coupling constants $r = r(u, v)$ and $g = g(u, v)$ of the lattice field depends on the coupling constants u and v of the continuum model and are given by non-Gaussian functional integral.

Denote the discretization transformation

$$(u, v) \to (r(u, v), g(u, v))$$

by $P(\alpha)$.

The transformation of the hierarchical renormalization group $r(\alpha)$ can be computed explicitly in the space of coupling constants of the hierarchical model and is given by the mapping

$$R(\alpha)(r, g) = (r', g')$$

$$r' = p^{\alpha-d} \left(\frac{(r+1)^2 - g}{(r+1)^2 - g/p^d}(r+1) - 1 \right),$$

$$v' = p^{2\alpha-3d} \left(\frac{(r+1)^2 - g}{(r+1)^2 - g/p^d} \right)^2 g,$$

Taking into account that $r(\alpha)$ is the discrete version of scaling transformation $S_{p^{-1}}(\alpha)$, we have

$$R(\alpha)P(\alpha) = P(\alpha)S(\alpha).$$

The mapping $S(\alpha)$ is given by the diagonal matrix whose eigenvalues are the eigenvalues of the differential of $R(\alpha)$ at the origin. Hence we can treat the mapping $P(\alpha)$ as a normalizing transformation to the mapping $R(\alpha)$ at the zero point an can find functional integral $P(\alpha)$ as a solution of the classical functional equation

For $\alpha > 3d/2$ both eigenvalues are more than 1, and we are in the domain where the classical Poincare theorem applies. According to to this theorem, the mapping $P(\alpha)$ can be expanded in a power series in u and v that converges for sufficiently small u and v provided α is a non-resonance value. The resonance values in the domain $3d/2 < \alpha < 2d$

are arranged in the discrete series

$$\alpha_k = (\frac{3}{2} + \frac{1}{2(2k-1)})d, \quad k = 1, 2, \ldots.$$

and exactly corresponds to the ultraviolet poles of p-adic Feynman amplitudes. If $d \leq \alpha \leq 3d/2$, the second eigenvalue is less then 1 and we are in the so-called Siegel domain. In that case any rational α is a resonance value and the convergence of the mapping $P(\alpha)$ requires some Diophantine condition.

The renormalization procedure can be defined as the mapping inverse to the normal mapping $P(\alpha)$. We can restore coupling constants of the continuum theory from the coupling constants of the discrete model using inverse map $P^{-1}(\alpha)$. We can embed locally $R(\alpha)$ in the continuous semigroup

$$R^t(\alpha) = P(\alpha) S^t(\alpha) P^{-1}(\alpha).$$

Coefficients of the vector field generating this semigroup are quantum field β-functions of this theory.

We can prove rigorously that discrete model is well defined for the whole plane of coupling constants and almost all values of α. But we can prove rigorously that continuum model is well defined only in some small neighborhood of trivial (zero) fixed point of renormalization group. In other words the continuum model is related with the discrete model as the normal form is related with the map.

For discrete models we also have diagram

$$\begin{array}{ccc} (Z^d, R) & \longrightarrow & (Z^d, \Gamma) \\ \downarrow & & \downarrow \\ (T_p^d, R) & \longrightarrow & (Q_p^d, \Gamma) \end{array}$$

and we discuss in the following right lower corner. We'll use also RG–transformation in the space of non–normalized Grassmann–valued "densities" of single spin "distribution"

$f(\psi^*; c_0, c_1, c_2) = c_0 + c_1(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + c_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$. Particularly, if $c_0 \neq 0$, we can write the density f in the exponential form $f(\psi^*; c_0, c_1, c_2) = c_0 \exp\{-L(\psi^*; r(c), g(c))\}$, where $c = (c_0, c_1, c_2)$, $r(c) = -c_1/c_0$, $g(c) = (c_1^2 - c_0c_2)/c_0^2$. If $c_0 = 0$, as, for example, in the case of Grassmann δ– function $\delta(\psi^*) = \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$, the exponential representation is impossible.

The RG–transformation in projective representation have the following form: $R(\alpha)(c_0, c_1, c_2) = (c_0', c_1', c_2')$,

$$c_0' = n^2(c_2 - 2c_1 + c_0)^{n-2}\left((c_1 - c_0)^2 + \frac{1}{n}(c_0c_2 - c_1^2)\right),$$

$$c_1' = \lambda n^2(c_2 - 2c_1 + c_0)^{n-2}\left((c_1 - c_0)(c_2 - c_1) + \frac{1}{n}(c_0c_2 - c_1^2)\right),$$

$$c_2' = \lambda^2 n^2(c_2 - 2c_1 + c_0)^{n-2}\left((c_2 - c_1)^2 + \frac{1}{n}(c_0c_2 - c_1^2)\right).$$

$\lambda_1 = p^{\alpha-d}$, $n = p^d$. If $c_2 - 2c_1 + c_0 \neq 0$, we can omit this factor.

The mapping $R(\alpha)$ is correctly defined as the mapping from 2-dimensional projective space to itself everywhere except the point (1,1,1) because $R(\alpha)(1,1,1) = (0,0,0)$ In the (r,g)-plane this point is given by (-1,0) and we call this point the singular point of the RG map.

RG–transformation in (r,g)–space have three finite fixed points, if $\alpha \neq 1$:

$$r_0(\alpha) \equiv 0 \quad g_0(\alpha) \equiv 0,$$

$$r_+(\alpha) = \frac{p^{d/2} - p^{\alpha-d}}{1 - p^{d/2}}, \quad g_+(\alpha) = \frac{r_+(\alpha)(1 + r_+(\alpha))^2}{1 + r_+(\alpha) + p^{-d/2}},\ \alpha \neq \frac{d}{2},$$

$$r_-(\alpha) = \frac{-p^{d/2} - p^{\alpha-d}}{1 + p^{d/2}}, \quad g_-(\alpha) = \frac{r_-(\alpha)(1 + r_-(\alpha))^2}{1 + r_-(\alpha) - p^{-d/2}}.$$

For $\alpha = d$ we have a whole line of fixed points $\{g = 0, r \neq -1\}$.

One can see, that Grassmann Fourier transform transposes the coefficients c_0, c_1, c_2 of the density $f(\bar{\eta}, \eta; c_0, c_1, c_2)$:

$$F_{\eta^* \to \xi^*}(f(\eta^*; c_0, c_1, c_2)) =$$

$$= \int \exp\{-(\bar{\xi}_1\eta_1 + \bar{\xi}_2\eta_2 + \xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2)\} f(\eta^*; c_0, c_1, c_2)\, d\eta_1\, d\bar{\eta}_1\, d\eta_2\, d$$

$$= f(\xi^*; c_2, c_1, c_0).$$

It is easy to verify commutation relation

$$FR(\alpha) = R(2d - \alpha)F.$$

Besides three finite fixed points there is infinitely far fixed point, which in c–space is described by the vector $(0, 0, 1)$, determining the Grassmann δ–function density $f(\psi^*; 0, 0, 1) = \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2$. It is possible to describe RG-invariant sets and behavior of stable RG-invariant curves. Let us consider upper half-plane $(r, g) : g > 0$. One can prove the following

Theorem. The part of stable RG–invariant curve γ_+, passing through the "+" FP for $\alpha > 3/2$ is given by equation $g = h_+(r; \alpha),\ 0 \leq r < \infty$, where $h_+(r; \alpha)$ is a smooth monotonically increasing function (in r). For $d < \alpha \leq 3d/2$ the same equation defines stable RG–invariant curve for the trivial FP. The part of stable RG–invariant curve

γ_-, passing through the "$-$"–FP for $\alpha > 1$ is given by the equation $g = h_-(r;\alpha)$ $-\infty < r < -1$, where $h_-(r;\alpha)$ is a smooth monotonically decreasing function. Moreover $h_+(r;\alpha) \to 0$, when $r \to 0$ and $h_+(r;\alpha) \to \infty$, when $r \to \infty$, $\quad h_-(r;\alpha) \to 0$, when $r \to -1$, $h_-(r;\alpha) \to \infty$, when $r \to -\infty$.

Let $\Omega_+ = \{(r,g) : 0 < r < \infty, 0 \le g < h_+(r;\alpha)\}$, $\Omega_- = \{(r,g) : -\infty < r < -1, 0 \le g < h_-(r;\alpha)\}$. We have proved, that domains Ω_+ and Ω_- are RG–invariant.

Let $\Omega = \{(r,g) : g > 0\} \setminus (\Omega_+ \cup \Omega_-)$ (the domain in the upper half–plane, bounded by the curve γ_+ on the right and by the curve γ_- on the left).

Here we use a computer graphics: every point of the upper half–plane is colored red (blue), if after some finite number of RG–iterations this point maps into $\Omega_+(\Omega_-)$. To obtain the global picture we use the same algorithm in c–

space. We identify the projective c–space with hemisphere $S = \{(c_0, c_1, c_2) : c_0^2 + c_1^2 + c_2^2 = 1,\ c_0 \geq 0\}$, where the opposite points of the boundary circle $c_1^2 + c_2^2 = 1$ must be identified also. Moreover, to obtain the planar picture, we use the orthogonal projection of S onto the disk $S_1 = \{(c_1, c_2) : c_1^2 + c_2^2 \leq 1\}$. The set of points attracting by the Ω_+ is described by the following way. There is a countable series of nonintersecting subsets $C, A(1), A(2), A(3),$ of the domain Ω. Let $A(0)$ denote the union of sets C and Ω_+. In turn, every of zones $A(i), i = 1, 2,$ has its own (satellite) countable series of nonintersecting subsets $A(i, j), j = 1, 2,$ and so on. If point belongs to zone $A(i_1, i_2, , i_k)$ and $i_1 > 1$, then after one RG-iteration it goes to zone $A(i_1 - 1, i_2, , i_k)$. If $i_1 = 1$ then it goes to $A(i_2, i_3, , i_k)$. Point from zone $A(i), i = 1, 2,$ after one RG-iteration goes to $A(i - 1)$.

$$A(i_1, i_2, , i_k) \to A(i_1 - 1, i_2, , i_k) \to ...$$

$$\to A(1, i_2, , i_k) \to A(i_2, , i_k) \to ...$$

$$\to A(i_3, i_4, , i_k) \to ... \to A(0).$$

Structure of the blue subsets $B(i_1, i_2, , i_k)$ of the points attracted by left domain Ω_- is analogues to the red subsets. All other points of Ω lie on the boundaries of zones $A(i_1, i_2, , i_k)$ and $B(i_1, i_2, , i_k)$. These boundaries are invariant curves of the map $R(\alpha)$ or its degree. Numerically it is found that there are cycles of RG-map of the order k for $k < 10$ and they lie on the boundaries of sets $A(k, k, k, ...)$ and $B(k, k, k, ...)$.

We describe several new interesting phenomena discovered in our model. Some of them can be generalized for the other corners of our diagram.

1. Interpretation of renormalization procedure as a normal form to the renormalization group transformation at zero fixed point. This interpretation is valid for the bosonic hierarchical model .

2. New branch of fixed points and cycles of renormalization group. It is possible to construct locally the non-Gaussian branch of fixed points which bifurcates from the Gaussian for models in all other corners of the diagram. Will be very interesting to find another branch of fixed points in the fermionic Euclidean model.

3.Commutative relation between renormalization group and Fourier transformations $F\,R(\alpha) = R(2d-\alpha)F$. This relation is true for p-adic and Euclidean case .

4. "+"-branch of fixed points lies in the lower half-plane for $d < \alpha \leq 3d/2$. As it follows from the property 3 the non-Gaussian branch of fixed points in the bosonic case is well defined for $\alpha \leq d/2$ and bifurcates from the fixed point at infinity which corresponds to constant (zero)random field . But what about $d < \alpha \leq 3d/2$?

5. Special role of $\alpha = d$. It follows from the previous commutative relation and the fact that all cycles go to the singular point (-1,0) when α tends to d.

6.Similarity of $(\alpha - 3d/2)$-expansions for critical exponents in p-adic and Euclidean bosonic models .

7.$(\alpha - 3d/2)$- and $(4 - d)$-expansions describe the same non-Gaussian fixed point at dimension $d = 3$. We have arguments for that is true in the bosonic hierarchical model .