p-Adic wavelets

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Wavelets on $\mathbb{R}$

Wavelet basis on $\mathbb{R}$: $\{2^{j/2}\psi(2^j x - n),\ \ j, n \in \mathbb{Z}\}$

A. Haar, 1910: $\psi(x) = \begin{cases} 1, & \text{if} \quad x \in (0, 1/2), \\ -1, & \text{if} \quad x \in (1/2, 1), \end{cases}$ $\quad \operatorname{supp} \psi \subset [0, 1].$

Y. Meyer, S. Mallat, 1988: multiresolution analysis (MRA)

Definition

A collection of closed spaces $V_j \subset L_2(\mathbb{R})$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* (*MRA*) in $L_2(\mathbb{R})$, if the following conditions (axioms) hold:

1. $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$; **2.** $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$; **3.** $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

4. $f \in V_j \iff f\left(2^{-j}\cdot\right) \in V_0$ for all $j \in \mathbb{Z}$;

5. there exists a function $\varphi \in V_0$ such that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

Each MRA generates an orthonormal wavelet basis for $L_2(\mathbb{R})$.

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Daubechies wavelets (which provided JPEG-2000): ψ is compactly supported, $\psi \in C^r(\mathbb{R})$.

The exist orthogonal wavelet bases which are not generated by an MRA.

Example:

$$\widehat{\psi} = \mathbb{1}_{[-4/7,-2/7]} + \mathbb{1}_{[2/7,3/7]} + \mathbb{1}_{[12/7,16/7]}$$

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Haar bases in different structures

W.C. Lang, 1996: Cantor group;

Yu.A. Farkov, 2008: Vilenkin group;

S.Kozyrev, 2002: group of p-adic numbers;

S.F.Lukomskii, 2009-2012: zero-dimension groups;

H.Aimar, A.Bernardis, B.Iaffei, 2007: a class of metric spaces;

S.Evdokimov, 2012: ring of adeles

Cantor group

An abelian locally compact group.
Elements are $x = \{x_k\}_{k=-\infty}^{\infty}$, where $x_k \in \{0, 1\}$, and $x_k \neq 0$ for only a finite number of negative k (interpreted as positive numbers: $x = x_{-N-1}2^N + \cdots + x_{-1}2^0 + x_0 2^{-1} + x_1 2^{-2} + \ldots$);
coordinate-wise mod 2 addition $x + y$;
dilation operator D takes $x = \{x_k\}_{k=-\infty}^{\infty}$ to $Dx = \{x_{k+1}\}_{k=-\infty}^{\infty}$ (interpreted as multiplication by 2).

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$$\psi(x) = \begin{cases} 1, & \text{if} \quad x = \{x_k\}_{k=0}^{\infty}, x_0 = 0, \\ -1, & \text{if} \quad x = \{x_k\}_{k=0}^{\infty}, x_0 = 1, \end{cases} \quad I = \{x = \{x_k\}_{k=-\infty}^{-1}\}.$$

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Why are Haar bases in various structures the same?

Answer: I. Ya. Novikov and M.S., Mathematical Notes, 91 (2012), no 5-6, 895-898

(Ω, Σ, μ)

Let $\mathbf{p} = \{p_j\}_{j=-\infty}^{\infty}$ be a sequence of integers, $p_j > 1$. Assume that there exist collections of sets Ω_{jn}, $n \in \mathbb{Z}_+$, which are mutually disjoint for each $j \in \mathbb{Z}$, and such that $\mu\Omega_{0n} = 1$ for every $n \in \mathbb{Z}_+$, $\Omega = \cup_n \Omega_{jn}$ for every $j \in \mathbb{Z}$, and each $\Omega_{j-1,n}$ is divided into p_j equimeasured subsets Ω_{j,n_k}, $n_k = n_k(n,j) \in \mathbb{Z}$.

Collection of such $\{\Omega_{jn}\}_{(j,n)}$ is called $(H, \mathbf{p})$-partition.

Haar bases generated by $(H, \mathbf{p})$-partition

Let (Ω, Σ, μ) be equipped with a topology τ such that Σ contains all open sets, the measure μ be regular, and $\Omega = \{\Omega_{jn}\}_{(j,n)}$ be $(H, \mathbf{p})$-partition.
Assume that for every $x \in \Omega$ and for every element U of the base of neighborhoods of a point x there exists Ω_{jn} containing x and contained in U.

Given pair (j, n), let $\Omega_{j,n} = \bigcup\limits_{k=0}^{p_{j+1}-1} \Omega_{j+1,n_k}$,

$$\psi_{jn}^{\nu} = C_{j+1} \sum_{k=0}^{p_{j+1}-1} h_{\nu k} \mathbb{1}_{\Omega_{j+1,n_k}}, \quad \nu = 1, \dots, p_{j+1} - 1,$$

where $h_{\nu k}$, $\nu, k = 0, \dots, p_{j+1} - 1$, are entries of a unitary matrix whose first row consists of elements equal to each other.
The system $\Psi_{j,n} := \bigcup\limits_{(j,n)} \{\psi_{jn}^{\nu}, \nu = 1, \dots, p_{j+1} - 1\}$ is a Haar basis for $L_2(\Omega)$

If τ is defined by a metric, then μ is regular. In this case the assumption of the theorem is satisfied whenever diameters of included sets Ω_{jn} containing x tend to zero as $j \to +\infty$.
If there exists a dilation operator $D : \Omega \to \Omega$ such that $D^{-1}\Omega_{jn} = \Omega_{j+1,n}$ for all $n \in \mathbb{Z}_+$, then we obtain Haar basis in traditional form
$\psi^{\nu}_{jk}(x) = C_j \psi^{\nu}_{0k}(D^j x)$, $k \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, $\nu = 1, \dots, p_{j+1} - 1$.

Similarly, an appropriate sequence of dilation operators $D_j : \Omega \to \Omega$ such that $D_j^{-1}\Omega_{jn} = \Omega_{j+1,n}$ for all $n \in \mathbb{Z}_+$, leads to Haar basis.

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For most Haar bases, it is easy to see that a natural dilation operator D (or a sequence of dilation operators D_j) provides an $(H, \mathbf{p})$-partition.

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S.Albeverio and S.Kozyrev, 2009: $\mathbb{Q}_p^d$, $D: \quad x \to Ax$, where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & p \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

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Slightly less trivial to see that the operator $D : x \to Ax$, where $D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, provides an $(H, \mathbf{p})$-partition for $\mathbb{Q}_p^2$.
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$\mathbb{Q}_p$ is the field of p-adic numbers

$B_\gamma(a) := \{x \in \mathbb{Q}_p : |x-a|_p \le p^\gamma\}$, $a \in \mathbb{Q}_p$, $\gamma \in \mathbb{Z}$.

$\mathbb{Z}_p := B_0(0)$ is the ring of p-adic integers;
$\mathbb{Z}_p := \Big\{x \in \mathbb{Q}_p : \quad x = [x] := x - \{x\}\Big\}$

$I_p := \Big\{x \in \mathbb{Q}_p : \quad x = \{x\}\Big\} \qquad (I_2 = \{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \dots\})$

I_p is a natural set of translations because

$$\mathbb{Q}_p = \bigcup_{a \in I_p} B_0(a) = \bigcup_{a \in I_p} (\mathbb{Z}_p + a)$$

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- We consider complex-valued function f defined on $\mathbb{Q}_p$.
- A function f defined on $\mathbb{Q}_p$ is called periodic if there exists $m \in \mathbb{Z}$ such that $f(x + p^m) = f(x)$ for every $x \in \mathbb{Q}_p$.
- $\mathcal{D}$ denotes the set of compactly supported periodic functions (so-called test functions). The space $\mathcal{D}$ is an analog of the Schwartz space in the real analysis.
- Since $\mathbb{Q}_p$ is a locally compact abelian group with a compact open subgroup, there exists a Haar measure dx on $\mathbb{Q}_p$, and the corresponding space $L_2(\mathbb{Q}_p)$.

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- $\chi_p(t) = e^{2\pi i t} := e^{2\pi i \{t\}_p}$ are the additive characters of $\mathbb{Q}_p$,
- The Fourier transform of $\varphi \in \mathcal{D}$ is defined by the formula

$$\widehat{\varphi}(\xi) := \int_{\mathbb{Q}_p} \chi_p(\xi \cdot x)\varphi(x)\, dx.$$

- One extends the Fourier transform onto $L_2(\mathbb{Q}_p)$ in the standard way.
- Interesting fact: $\varphi = \mathbb{1}_{B_0(0)} = \widehat{\varphi}$
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- One extends the Fourier transform onto $L_2(\mathbb{Q}_p)$ in the standard way.
- Interesting fact: $\varphi = \mathbb{1}_{B_0(0)} = \widehat{\varphi}$
- $\varphi \in \mathcal{D}$ if and only if both the functions $\varphi, \widehat{\varphi}$ are compactly supported.
- φ is p^m-periodic if and only if $\operatorname{supp} \widehat{\varphi} \subset B_m(0)$.

p-Adic Haar basis

$$p^{-j/2}\psi^{(\nu)}(p^j x - a),\ j \in \mathbb{Z},\ a \in I_p,\ \nu = 1, \dots, p-1,$$

where $\psi^{(\nu)}(x) = \chi_p(\nu x/p)\mathbb{1}_{B_0(0)}(x)$.

Kozyrev, S.V. *Wavelet analysis as a p-adic spectral analysis*, Izvestia Akademii Nauk, Seria Math., **66** (2002), no. 2, 149–158.

$(H, \mathbf{p})$-partition

$\Omega = \mathbb{Q}_p$, $p_j = p$;

$\Omega_{0n} = B_0(a_n) = \mathbb{Z}_p + a_n$, where $\{a_n\}_{n=1}^{\infty} = I_p$;

$\Omega_{jn} = D^{-1}\Omega_{j-1,n}$, where $D^{-1}: \quad x \longrightarrow px$;

$\{h_{\nu k}\}_{\nu,k=0}^{p-1} = \{e^{2\pi i \nu k}\}_{\nu,k=0}^{p-1}$.

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Definition

A collection of closed spaces $V_j \subset L^2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$, is called a multiresolution analysis (MRA) in $L^2(\mathbb{Q}_p)$ if the following axioms hold

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{Q}_p)$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(d) $f(\cdot) \in V_j \iff f(p^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(e) there exists $\varphi \in V_0$ such that $V_0 = \overline{\text{span}\,\{\varphi(x-a),\ a \in I_p\}}$.

The function φ from axiom (e) is called scaling. If, moreover, the system $\{\varphi(x-a), a \in I_p\}$ is an orthonormal basis for V_0, then φ is called orthogonal scaling function.

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Theorem

There exists an MRA generated by an orthogonal scaling function.

V. M. Shelkovich and M. S., *p-Adic Haar multiresolution analysis and pseudo-differential operators*, J. Fourier Analysis and Appl., 15 (2009), N 3, 366-393

This MRA (Haar MRA) is generated by the scaling function is $\varphi = \mathbb{1}_{B_0(0)}$.

Define the Haar wavelet spaces W_j, $j \in \mathbb{Z}$, by $V_j \oplus W_j = V_{j+1}$. Due to axioms (a), (b), (c), $L^2(\mathbb{Q}_p) = \bigoplus_{j\in\mathbb{Z}} W_j$.

The functions $\psi^{(\nu)}(x) = \chi_p(\nu x/p)\mathbb{1}_{B_0(0)}(x)$ are in W_0, and their I_p-translations form an orthonormal basis for W_0.
It follows that $p^{-j/2}\psi^{(\nu)}(p^j x - a)$, $j \in \mathbb{Z}$, $a \in I_p$, $\nu = 1, \dots, p-1$, is an orthonormal basis for $L^2(\mathbb{Q}_p)$.

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How to construct another MRA?

Let φ be a scaling function for a MRA. It follows from axiom (a) that $V_0 \subset V_1$. **In the real setting** the relation $V_0 \subset V_1$ holds if and only if φ is refinable, i.e. φ satisfies a refinement equations: $\varphi(x) = \sum_{n\in\mathbb{Z}} h_n \varphi(2x - n)$.

What happens in p-adics?

$\varphi(x) = \sum_{a\in I_p} \beta_a \varphi(p^{-1}x - a)$, – p-adic refinement equation.

If $\varphi \in L^2(\mathbb{Q}_p)$ is a scaling function and $\operatorname{supp}\varphi \subset B_N(0)$, $N \geq 0$, then φ is refinable and its refinement equation is

$$\varphi(x) = \sum_{k=0}^{p^{N+1}-1} h_k \varphi\Big(\frac{x}{p} - \frac{k}{p^{N+1}}\Big).$$

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Let φ be refinable (we consider only scaling functions $\varphi \in \mathcal{D}$).
Can we hope that φ generates a MRA?

For axiom (a), every $\varphi(x-b)$, $b \in I_p$, should be decomposed with respect to the system $\{p^{1/2}\varphi(p^{-1}x-a), a \in I_p\}$.

Generally speaking, we cannot state that axiom (a) of the definition of MRA is fulfilled because I_p is not a group.
For example, $a = \frac{3}{4} \in I_2$, $b = \frac{1}{2} \in I_2$, $a + b \notin I_2$

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If $\operatorname{supp}\widehat{\varphi} \subset B_0(0) \iff \varphi(x+1) = \varphi(x)\ \forall x \in \mathbb{Q}_p$, then axiom (a) of the definition of MRA is fulfilled because for all $a, b \in I_p$ there exists $c \in I_p$ such that $a + b \equiv c \pmod 1$.

Theorem

Let $m_0(\xi) = \frac{1}{p}\sum_{k=0}^{p^{N+1}-1} \beta_k \chi_p(k\xi)$, $m_0(0) = 1$, $\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{p^{N-j}}\right)$. If $m_0\left(\frac{k}{p^{N+1}}\right) = 0$ for all $k = 1, \dots, p^{N+1} - 1$ not divisible by p, then $\operatorname{supp}\widehat{\varphi} \subset B_0(0)$. If, furthermore, $\left|m_0\left(\frac{k}{p^{N+1}}\right)\right| = 1$ for all $k = 1, \dots, p^{N+1} - 1$ divisible by p, then $\{\varphi(x - a) : a \in I_p\}$ is an orthonormal system, i.e. φ is an orthogonal scaling function generating a MRA.

A.Yu. Khrennikov, V.M. Shelkovich and M. S., *p-Adic refinable functions and MRA-based wavelets*, J. Approx. Theory, 161 (2009) 226-238

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Thus there exists a wide class of refinable functions φ with *supp* $\widehat{\varphi} \subset B_0(0)$ generating a MRA.

However, **in contrast to the real setting**,

Theorem

There exists a unique MRA generated by an orthogonal scaling function $\varphi \in \mathcal{D}$, $\mathrm{supp}\,\widehat{\varphi} \subset B_0(0)$. *It is Haar MRA.*

Is it possible a function φ with $\operatorname{supp}\widehat{\varphi} \not\subset B_0(0)$ to be an orthogonal scaling function?

Theorem

There exist MRAs with scaling functions φ such that $\operatorname{supp}\widehat{\varphi} \not\subset B_0(0)$.

Theorem

Let $\varphi \in \mathcal{D}$ be an orthogonal scaling function and $\widehat{\varphi}(0) \neq 0$. Then $\operatorname{supp}\widehat{\varphi} \subset B_0(0)$.

Corollary

There exists a unique MRA generated by an orthogonal scaling function. It is Haar MRA.

S. Albeverio, S. Evdokimov and M. S. *p-Adic multiresolution analysis and wavelet frames* J. Fourier Anal. Appl., 16 (2010), No. 5, 693-714

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So, it is not possible to construct orthogonal wavelets based on a MRA generated by a scaling function $\varphi \in \mathcal{D}$ with $\operatorname{supp}\widehat{\varphi} \not\subset B_0(0)$.

Are there non-Haar p-adic orthogonal wavelet bases?

S. Evdokimov and M. S. *Description of p-adic orthogonal wavelet bases generated by periodic functions.* (prepared for publication)

What means "A wavelet basis is generated by the Haar MRA"?

Haar MRA $\{V_j\}_{j\in\mathbb{Z}}$, $W_j := V_{j+1} \ominus V_j$, $L_2(\mathbb{Q}_p) = \bigoplus_{j\in\mathbb{Z}} W_j$

A collection of functions $\psi^{(\nu)} \in W_0$, $\nu = 1, \dots, p-1$, such that $\{\psi^{(\nu)}(x-a), a \in I_p, \nu = 1, \dots, p-1\}$ is an orthonormal basis for W_0 is called a standard set of Haar wavelet functions.

The corresponding wavelet system

$$\{p^{j/2}\psi^{(\nu)}(p^{-j}x - a), a \in I_p, j \in \mathbb{Z}, \nu = 1, \dots, p-1\}$$

is an orthonormal basis for $L_2(\mathbb{Q}_p)$. Such bases are called standard Haar bases.

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$p=2$, $\psi(x)=\chi_2(\nu x/2)\mathbb{1}_{B_0(0)}(x)$, $\psi_{ja}(x):=2^{j/2}\psi(2^{-j}x-a)$

$\{\psi\}$ is a standard set of Haar wavelet functions.

$\{\psi_{ja}, a\in I_p, j\in\mathbb{Z}\}$ is a standard Haar basis

$$\psi^{(1)}(x)=\sqrt{2}\psi(x/2),\quad \psi^{(2)}(x)=\sqrt{2}\psi((x-1)/2).$$

$$\{\psi_{ja}^{(\nu)}, a\in I_p, j\in\mathbb{Z}, \nu=1,2\}=\{\psi_{ja}, a\in I_p, j\in\mathbb{Z}\}$$

$\{\psi^{(1)},\psi^{(2)}\}$ is not a standard set of Haar wavelet functions.
$\{\psi_{ja}^{(1)},\psi_{ja}^{(2)}, a\in I_p, j\in\mathbb{Z}\}$ is a standard Haar basis

$$\psi^{(2,1)}(x)=\sqrt{2}\psi^{(2)}(x/2),\quad \psi^{(2,2)}(x)=\sqrt{2}\psi^{(2)}((x-1)/2),$$

$\{\psi_{ja}^{(1)},\psi_{ja}^{(2,1)},\psi_{ja}^{(2,2)}, a\in I_p, j\in\mathbb{Z}\}$ is a standard Haar basis.

$$\widetilde{\psi}^1 = \frac{1}{\sqrt{2}}(\psi^{(1)} + \psi^{(2,1)}), \quad \widetilde{\psi}^2 = \frac{1}{\sqrt{2}}(\psi^{(1)} - \psi^{(2,1)}), \quad \widetilde{\psi}^3 = \psi^{(2,2)}.$$

$\{\widetilde{\psi}^1_{ja}, \widetilde{\psi}^1_{ja}, \widetilde{\psi}^1_{ja}, a \in I_p, j \in \mathbb{Z}\}$ is not a standard Haar basis.

The vector-function $\widetilde{\Psi} = (\widetilde{\psi}^1, \widetilde{\psi}^2, \widetilde{\psi}^3)^T$ is unitary equivalent to a vector-function generating a standard Haar basis.

$$\widetilde{\psi}^{(1,1)}(x) = \sqrt{2}\widetilde{\psi}^{(1)}(x/2), \quad \widetilde{\psi}^{(1,2)}(x) = \sqrt{2}\widetilde{\psi}^{(1)}((x-1)/2).$$

The vector-function $\widetilde{\Psi}' = (\widetilde{\psi}^{(1,1)}, \widetilde{\psi}^{(1,2)}, \widetilde{\psi}^{(2)}, \widetilde{\psi}^{(3)})^T$ does not generate a standard Haar basis, and $\widetilde{\Psi}'$ is not unitary equivalent to a vector-function generating a standard Haar basis.

Let us say that a basis obtained in this way is a "damaged " Haar basis.

All non-standard orthogonal p-adic wavelet bases we saw in the literature are "damaged " Haar bases.

J.J. Benedetto, and R.L. Benedetto, A wavelet theory for local fields and related groups, The Journal of Geometric Analysis **3** (2004) 423–456

Khrennikov, A. Yu. and Shelkovich, V. M. Non-Haar p-adic wavelets and pseudodifferential operators, (Russian) Dokl. Akad. Nauk 418 (2008), no. 2, 167–170

Definition

Two vector-functions Ψ, Ψ' generating orthonormal wavelet bases is said to be wavelet equivalent if there exist vector-functions $\Psi^{(0)}, \dots, \Psi^{(N)}$ such that $\Psi^{(0)} = \Psi$, $\Psi^{(N)} = \Psi'$, and for every $j > 0$ either $\Psi^{(j)}$ is unitary equivalent to $\Psi^{(j-1)}$ or $\Psi^{(j)}$ and $\Psi^{(j-1)}$ generate the same orthonormal wavelet bases.

Theorem

Any periodic vector-function generating orthonormal wavelet bases is wavelet equivalent to a standard set of Haar wavelet functions.

Theorem

There exists a vector-function generating orthonormal wavelet bases which is not a "damaged " Haar bases.

Theorem

If periodic functions $\psi^{(1)}, \dots, \psi^{(r)}$ generate orthonormal wavelet bases, then r is divisible by $p-1$, in particular, $r \geq p-1$.

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Theorem

If a wavelet system generated by p^m-periodic functions $\psi^{(1)},\dots,\psi^{(r)}$, $m\in\mathbb{Z}_+$, is orthonormal, then $r\le(p-1)p^{m-1}$. In particular, if a function ψ is 1-periodic, then the wavelet system generated by ψ cannot be orthogonal.

Theorem

If functions $\psi^{(1)},\dots,\psi^{(r)}\in W_m$, $m\in\mathbb{Z}_+$, generate orthonormal wavelet bases, then $r=(p-1)p^m$.

Theorem

If functions $\psi^{(1)},\dots,\psi^{(r)}$ generate orthonormal wavelet bases, $\psi^{(\nu)}\in W_{j_\nu}$, then $\sum\limits_{\nu=1}^{r}p^{m-j_\nu}=(p-1)p^m$.

Biorthogonal wavelets on $\mathbb{R}$

To construct biorthogonal wavelets on $\mathbb{R}$ one starts with two dual MRAs $\{V_j\}_{j\in\mathbb{Z}}$, $\{\widetilde{V}_j\}_{j\in\mathbb{Z}}$ generated by scaling functions φ, $\widetilde{\varphi}$ whose integer translations $\varphi(\cdot - n)$, $\widetilde{\varphi}(\cdot - n)$ are biorthogonal and form Riesz bases for V_0, $\widetilde{V}_0$ respectively (instead of the requirement: $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for V_0.)

This leads to the construction of biorthogonal wavelets $\{2^{j/2}\psi(2^j x - n),\ \ j, n \in \mathbb{Z}\}$, $\{2^{j/2}\widetilde{\psi}(2^j x - n),\ \ j, n \in \mathbb{Z}\}$

Theorem

A function $\varphi \in \mathcal{D}$, $\operatorname{supp}\varphi \subset B_N(0)$, $\operatorname{supp}\widehat{\varphi} \subset B_M(0)$, $N, M \geq 0$, $\widehat{\varphi}(0) \neq 0$, generates a MRA if and only if
(1) φ is refinable;
(2) there exist at least $p^{M+N} - p^N$ integers n such that $0 \leq n < p^{M+N}$ and $\widehat{\varphi}\left(\frac{n}{p^M}\right) = 0$.

S. Albeverio, S. Evdokimov and M. S. *p-Adic multiresolution analysis and wavelet frames*, J. Fourier Anal. Appl., 16 (2010), No. 5, 693-714

Is it possible to construct non-Haar MRA-based biorthogonal wavelets?

E. King and M. S. *On p-adic biorthogonal wavelet bases.* (in preparation)

There exist dual pairs of non-Haar MRAs $\{V_j\}_{j\in\mathbb{Z}}$, $\{\widetilde{V}_j\}_{j\in\mathbb{Z}}$ generated respectively by scaling functions φ, $\widetilde{\varphi}$ whose I_p-translations $\varphi(\cdot - a)$, $\widetilde{\varphi}(\cdot - a)$ are biorthogonal.

Theorem

If biorthogonal p-adic wavelet systems $\{p^{j/2}\psi^{(\nu)}(p^{-j}x - a), a \in I_p, j \in \mathbb{Z}, \nu = 1, \dots, p-1\}$, $\{p^{j/2}\widetilde{\psi}^{(\nu)}(p^{-j}x - a), a \in I_p, j \in \mathbb{Z}, \nu = 1, \dots, p-1\}$ *are generated by dual MRAs* $\{V_j\}_{j\in\mathbb{Z}}$, $\{\widetilde{V}_j\}_{j\in\mathbb{Z}}$ *, then each of MRAs is the Haar MRA.*

Is it possible to construct MRA-based non-orthogonal wavelets generated by a scaling function φ with $\mathrm{supp}\,\widehat{\varphi} \not\subset B_0(0)$?

An infinite family of functions $\varphi_{M,N} \in \mathcal{D}$ with $\mathrm{supp}\,\widehat{\varphi}_{M,N} \subset B_M(0)$, $\mathrm{supp}\,\widehat{\varphi}_{M,N} \not\subset B_0(0)$ and the corresponding wavelet functions $\psi^{(\nu)}_{M,N}$, $\nu = 1, \dots, p-1$, was constructed explicitly.

Theorem

For integers $M, N \geq 0$, the function $\varphi_{M,N}$ generates an MRA if and only if $M \leq \frac{p^N-1}{p-1} - N$, Moreover, in this case, the functions $p^{-j/2}\psi^{(\nu)}_{M,N}(p^j x - a)$, $j \in \mathbb{Z}, a \in I_p, \nu = 1, \dots, p-1$, form a Riesz basis for $L_2(\mathbb{Q}_p)$ if and only if $M = \frac{p^N-1}{p-1} - N$.

S. Albeverio, S. Evdokimov, and M. Skopina, p-Adic Nonorthogonal Wavelet

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Thank you for your attention!