Adeles as a metric space and parabolic type equations on adeles

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p-adic absolute values

Let p is a prime number.

Each rational number can be represented as

$$x = p^{\gamma}\frac{m}{n}, \qquad \text{where } \gamma, m, n \in \mathbb{Z}, \quad (m, n) = 1.$$

p-adic Absolute Value

$$|x|_p = p^{-\gamma}$$

Denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. The elements of $\mathbb{Q}$ are called ***p*-adic numbers**.

Strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}$$

Elements of the p-adic analysis

Consider complex-valued functions over the p-adic field

$$f : \mathbb{Q}_p \to \mathbb{C}.$$

There is a Haar measure $d_p x$ on $\mathbb{Q}_p$, i.e. an integration is possible.

However, there is no classical derivative,

$$\lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} \qquad \text{is meaningless!}$$

Vladimirov's (1988) **pseudo-differential operator** is used instead

$$D^\alpha f = F^{-1} \circ |\xi|_p^\alpha \circ F[f],$$

where $F[f]$ is the Fourier transform.

Which p to use?

Yuri Manin [Y. I. Manin, Reflections on arithmetical physics, 1989.] conjectured that the world is adelic.

> *On the fundamental level our world is neither real, nor p-adic, it is adelic. For some reasons, reflecting the physical nature of our kind of living matter (e.g., the fact that we are build of massive particles), we tend to project the adelic picture onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate most important things arithmetically.*
>
> *The relation between "real" and "arithmetical" pictures of the world is that of complementary, like the relation between conjugate observables in quantum mechanics.*

There are two different generalizations of the one dimension Vladimirov operator onto n dimensional case. One is also called a Vladimirov operator and it is an operator with a symbol $|\xi_1|_p^{\alpha_1} \times |\xi_2|_p^{\alpha_1} \times \ldots \times |\xi_n|_p^{\alpha_1}$. The adelic analogue of this operator is known [Volovich, Radyno, Khrennirov, 2003], [Khrennikov, Kosyak, Shelkovich, 2012].

Second is the Taibleson operator

$$D_n^{\alpha} f = F^{-1} \circ \|\xi\|_p^{\alpha} \circ F[f],$$

where

$$\|\xi\|_p = \max\left\{|\xi_1|_p, \ldots, |\xi_n|_p\right\}$$

is the p-adic distance.

How to generalize the Taibleson pseudodifferential operator for adelic spaces?

The ring of adeles

The **ring of adeles** $\mathbb{A}$ is defined by

$$\mathbb{A} = \left\{(x_\infty, x_2, x_3, \ldots) : x_p \in \mathbb{Q}_p, \text{ and } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\right\}.$$

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$\mathbb{A}$ is a locally compact topological ring, a base for the topology consists of all the sets of the form

$$U \times \prod_{p \notin S} \mathbb{Z}_p,$$

where S is any finite set of primes containing ∞, and U is any open subset in $\prod_{p\in S} \mathbb{Q}_p$.

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The **ring of finite adeles** $\mathbb{A}_f$ is defined by

$$\mathbb{A}_f = \left\{(x_2, x_3, \ldots) : x_p \in \mathbb{Q}_p, \text{ and } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\right\}.$$

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- On adeles $dx_{\mathbb{A}} = dx_\infty \cdot dx_{\mathbb{A}_f}$.
- An adelic function is said to be *test* or *Bruhat-Schwartz* if it can be expressed as a finite linear combination, with complex coefficients, of factorizable functions

$$f = \prod_{p \leq \infty} f_p,$$

where

(A1) $f_\infty \in \mathcal{S}(\mathbb{R})$;

(A2) $f_p \in \mathcal{S}(\mathbb{Q}_p)$ for $p < \infty$;

(A3) f_p is the characteristic function of $\mathbb{Z}_p$ for all but finitely many $p < \infty$.

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- The Fourier transform of a factorizable adelic Bruhat-Schwartz function is

$$\widehat{f}(\xi) = \prod_{p \le \infty} \int_{\mathbb{Q}_p} f_p(x_p)\, \chi(-x_p \xi_p)\, dx_p.$$

Desired properties of an adelic metric

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Desired properties of a metric:

- the ring of adeles should be a complete metric space;
- the metric should induce the same topology;
- balls with respect to the metric should be compact sets of finite non-zero volumes;
- the metric should agree with the Fourier transform, i.e. the Fourier transform of a radial function should be a radial function;
- the space of Bruhat-Schwartz test functions should have a simple characterization in the metric.

First try

Consider a function

$$\|x\|_1 := \max_p |x_p|_p, \qquad x \in \mathbb{A}_f.$$

It induces a metric by the formula

$$\rho_1(x, y) = \|x - y\|_1, \qquad x, y \in \mathbb{A}_f.$$

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However, it does not induce the same topology. Consider the sequence

$$x^{(n)} = (\underbrace{0,0,\ldots,0}_{n},1,0,0,\ldots) \in \mathbb{A}_f, \qquad n \in \mathbb{N}.$$

Then $x^{(n)} \to (0,0,\ldots)$ in the topology of $\mathbb{A}_f$, however

$$\rho_1\left(x^{(n)},(0,0,\ldots)\right) = 1 \not\to 0, \quad n \to \infty.$$

Second try

Consider a function

$$\|x\|_2 := \max_p \frac{|x_p|_p}{p}, \qquad x \in \mathbb{A}_f$$

and the induced metric

$$\rho_2(x,y) = \|x-y\|_2, \qquad x, y \in \mathbb{A}_f.$$

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This second metric induces the same topology and the ring of finite adeles becomes a complete metric space with respect to this metric.

However, balls of radiuses larger than 1 are of infinite volumes and not compact in this metric; the Fourier transform does not transform a radial function $f = f(\|x\|_2)$ into a radial function.

Finally, the adelic metric

Consider a function

$$\|x\| := \begin{cases} \|x\|_1 & \text{if } x \notin \prod_p \mathbb{Z}_p, \\ \|x\|_2 & \text{if } x \in \prod_p \mathbb{Z}_p, \end{cases} \qquad x \in \mathbb{A}_f$$

and the induced metric $\rho(x, y) = \|x - y\|, \quad x, y \in \mathbb{A}_f$.

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and the induced metric $\rho(x,y) = \|x-y\|, \quad x,y \in \mathbb{A}_f$.

Theorem

The topology on $\mathbb{A}_f$ is metrizable, the metric is given by the function ρ. Furthermore, $(\mathbb{A}_f, \rho)$ is a complete non-Archimedean metric space.

Different representation and range of the metric

The function $\|\cdot\|$ can be also represented as

$$\|x\| = \max_p p^{-[[\mathrm{ord}_p(x_p)]]}, \qquad x \in \mathbb{A}_f \setminus \{0\},$$

where

$$[[t]] := \begin{cases} [t] & \text{if } t \geq 0 \\ [t]+1 & \text{if } t < 0, \end{cases}$$

here $[\cdot]$ denotes the integer part function.

Proposition

The range of values of the function ρ coincides with the set

$$\{0\} \cup \left\{p^j : p \text{ is prime}, j \in \mathbb{Z} \setminus \{0\}\right\}.$$

Some auxiliary functions

Given a positive real number x, we define

$$\Phi(x) = \prod_p p^{[[\log_p x]]},$$

If $x \geq 2$, $\Phi(x)$ coincides with the exponential of the second Chebyshev function

$$\psi(x) = \sum_p [\log_p x] \ln p = \sum_{p^k \leq x} \ln p.$$

For any prime number p and any $j \in \mathbb{Z} \setminus \{0\}$,

$$\Phi(p^{-j}) = \frac{p}{\Phi(p^j)}.$$

Examples:

$$\Phi(1) = 1, \quad \Phi(2) = 2, \quad \Phi(3) = 6, \quad \Phi(4) = 12, \quad \Phi(1/2) = 1.$$

Some auxiliary functions

We introduce the ordering on powers of prime numbers defining the **next** and **previous non-zero power of a prime operators** as

$$n_+ = \min\{p^\beta : n < p^\beta,\ p \text{ prime},\ \beta \in \mathbb{Z}\setminus\{0\}\},$$
$$n_- = \max\{p^\beta : p^\beta < n,\ p \text{ prime},\ \beta \in \mathbb{Z}\setminus\{0\}\}.$$

Examples:

$$8_+ = 9,\quad 9_+ = 11,\quad 11_+ = 13,\quad 13_+ = 16,\quad 16_+ = 17,$$
$$3_- = 2,\quad 2_- = 1/2,\quad 1_- = 1/2,\quad 1/2_- = 1/3,\quad 1/3_- = 1/4.$$

The following relations hold for any $n = p^j$, p is a prime and $j \in \mathbb{Z}\setminus\{0\}$

$$(n_-)_+ = n, \qquad (n_+)^{-1} = (n^{-1})_-,$$
$$(n_+)_- = n, \qquad (n_-)^{-1} = (n^{-1})_+,$$
$$\Phi\left((p^j)_-\right) = \frac{\Phi(p^j)}{p}.$$

Proposition

The the adelic ball $B_r := B_r(0)$ is a compact set and its volume is given by

$$\operatorname{vol}(B_r) = \Phi(r).$$

The the adelic sphere $S_r := S_r(0)$ is a compact set and its volume is given by

$$\operatorname{vol}(S_r) = \Phi(r) - \Phi(r_-).$$

Fourier transform and radial functions

Theorem

Let $f = f(\|\xi\|) : \mathbb{A}_f \to \mathbb{C}$ *be a radial function in* $L^1(\mathbb{A}_f)$. *Then for any* $x \in \mathbb{A}_f$

$$\check{f}(x) := \left(\mathcal{F}^{-1}_{\xi\to x} f\right)(x) = \sum_{q^j < \|x\|^{-1}} \Phi\left(q^j\right)\left(f(q^j) - f(q^j_+)\right),$$

where q^j *runs through all non-zero powers of prime numbers.*

Corollary

Let f be a characteristic function of a ball, i.e.,

$$f = \mathbf{1}_{B_r}(x), \qquad r \in \{p^j : p \text{ is prime}, \ j \in \mathbb{Z} \setminus \{0\}\}.$$

Then

$$\widehat{f}(\xi) = \Phi(r)\mathbf{1}_{B_R}(\xi), \quad R = (r^{-1})_-.$$

Corollary

Let f be a real-valued non-increasing radial function, i.e. $f = f(\|\xi\|)$ and $f(\xi) \ge f(\zeta)$ for any $\xi, \zeta \in \mathbb{A}_f$ satisfying $\|\xi\| \le \|\zeta\|$. Then

$$\check{f}(x) := \big(\mathcal{F}^{-1}_{\xi\to x} f\big)(x) \ge 0 \qquad \text{for any } x \in \mathbb{A}_f.$$

Bruhat-Schwartz test functions

Definition

A function f is locally constant if for any $x \in \mathbb{A}_f$ there exists a constant $\ell(x) > 0$ such that $f(x+y) = f(x)$ for any $y \in B_{\ell(x)}(0)$.

Definition

Let f be a non-zero compactly supported function. We define the parameter of constancy ℓ of f as the largest non-zero integer power of a prime number such that

$$f(x+y) = f(x) \qquad \text{for any } x \in \mathbb{A}_f, \ y \in B_\ell(0).$$

Proposition

The function f is a Bruhat-Schwartz test function if and only if it is locally constant with compact support.

Metric structure on adeles

Any $x \in \mathbb{A}$ can be written uniquely as

$$x = (x_\infty, x_f) \in \mathbb{R} \times \mathbb{A}_f = \mathbb{A}.$$

Set for $x, y \in \mathbb{A}$

$$\rho_{\mathbb{A}}(x, y) := |x_\infty - y_\infty|_\infty + \rho(x_f, y_f).$$

Proposition

The restricted product topology on $\mathbb{A}$ is metrizable, a metric is given by $\rho_{\mathbb{A}}$. Furthermore, $(\mathbb{A}, \rho_{\mathbb{A}})$ is a complete metric space and $(\mathbb{A}, \rho_{\mathbb{A}})$ as a topological space is homeomorphic to $(\mathbb{R}, |\cdot|_\infty) \times (\mathbb{A}_f, \rho)$.

Taibleson pseudodifferential operator on finite adeles

We introduce the Lizorkin space of the second kind:

$$\mathcal{L}_0(\mathbb{A}_f) := \mathcal{L}_0 = \left\{ f \in \mathcal{S}(\mathbb{A}_f) : \widehat{f}(0) = 0 \right\}.$$

and define for $\gamma > 0$ an operator

$$\left(D^\gamma f\right)(x) = \mathcal{F}^{-1}_{\xi \to x}\left(\|\xi\|^\gamma \mathcal{F}_{x\to\xi} f \right), \qquad f \in \mathcal{L}_0(\mathbb{A}_f).$$

Lemma

(i) $D^\gamma \mathcal{L}_0 = \mathcal{L}_0$ *for* $\gamma > 0$.

(ii) $f \in \mathcal{L}_0$ *if and only if* $f \in \mathcal{S}$ *and* $\int_{\mathbb{A}_f} f(x)\, dx_{\mathbb{A}_f} = 0$.

(iii) $\mathcal{L}_0$ *is dense in* $\mathcal{S}$ *with respect to the* L^2*-norm.*

(iv) $\mathcal{L}_0$ *is dense in* $L^2(\mathbb{A}_f)$.

The operator D^γ with the domain $\mathcal{D}(D^\gamma) = \mathcal{L}_0(\mathbb{A}_f)$ is symmetric and is essentially self-adjoint.

Lemma

The closure of the operator D^γ, $\gamma > 0$ (let us denote it by D^γ again) with the domain

$$\mathcal{D}\left(D^\gamma\right) := \left\{ f \in L^2\left(\mathbb{A}_f\right) : \|\xi\|^\gamma \widehat{f} \in L^2\left(\mathbb{A}_f\right) \right\}$$

is a self-adjoint operator. Moreover,

- (i) $D^\gamma \geq 0$;
- (ii) $\sigma(D^\gamma) = \left\{ p^{\gamma j} : p \text{ is a prime}, \ j \in \mathbb{Z} \setminus \{0\} \right\} \cup \{0\}$;
- (iii) $-D^\gamma$ *is the infinitesimal generator of a contraction C_0 semigroup $\left(\mathcal{T}(t)\right)_{t \geq 0}$. Moreover, the semigroup $\left(\mathcal{T}(t)\right)_{t \geq 0}$ is bounded holomorphic (or analytic) with angle $\pi/2$.*

Taibleson pseudodifferential operator on adeles

We introduce the Lizorkin space of the second kind

$$\mathcal{L}_0(\mathbb{A}) = \mathcal{L}_0(\mathbb{R}) \otimes \mathcal{L}_0(\mathbb{A}_f),$$

and define for $\alpha, \beta > 0$ an operator

$$\left(D^{\alpha,\beta} h\right)(x) = \mathcal{F}^{-1}_{\xi \to x} \left(\left(|\xi_\infty|_\infty^\beta + \|\xi_f\|^\alpha \right) \mathcal{F}_{x \to \xi} h \right), \qquad h \in \mathcal{L}_0(\mathbb{A}).$$

The Lizorkin space $\mathcal{L}_0(\mathbb{A})$ is invariant for $D^{\alpha,\beta}$.

Similarly, an operator $D^{\alpha,\beta}$ is essentially self-adjoint.

Lemma

The closure of the operator $D^{\alpha,\beta}$, $\alpha,\beta>0$ *(let us denote it by* $D^{\alpha,\beta}$ *again) with domain*

$$\mathcal{D}\left(D^{\alpha,\beta}\right) := \left\{f\in L^2\left(\mathbb{A}\right) : \left(|\xi_\infty|_\infty^\beta + \|\xi\|^\alpha\right)\widehat{f}\in L^2\left(\mathbb{A}\right)\right\}$$

is a self-adjoint operator. Moreover, the following assertions hold:

(i) $D^{\alpha,\beta}\geq 0$;

(ii) $\sigma(D^{\alpha,\beta})=[0,\infty)$;

(iii) $-D^{\alpha,\beta}$ *is the infinitesimal generator of a contraction* C_0 *semigroup* $\left(\mathcal{T}_{\alpha,\beta}(t)\right)_{t\geq 0}$. *Moreover, the semigroup* $\left(\mathcal{T}_{\alpha,\beta}(t)\right)_{t\geq 0}$ *is bounded holomorphic (or analytic) with angle* $\pi/2$.

Parabolic type equation and Cauchy problems

Consider for $\alpha > 1$ the Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + D^{\alpha} u(x,t) = 0, & x \in \mathbb{A}_f,\ t \in [0,+\infty)\,, \\ u(x,0) = u_0(x), & u_0(x) \in \mathcal{D}(D^{\alpha}). \end{cases} \tag{1}$$

together with inhomogeneous problem.

The operator $-D^{\alpha}$ generates a C_0 semigroup.
Hence the solution of (1) is given by

$$u(x,t) = \mathcal{T}(t) u_0(x), \qquad t \geq 0$$

and the problem is to describe the semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$.

We define *the adelic heat kernel* on $\mathbb{A}_f$ as

$$Z(x,t) = \int_{\mathbb{A}_f} \chi(\xi \cdot x)\, e^{-t\|\xi\|^\alpha} d\xi_{\mathbb{A}_f}, \quad x \in \mathbb{A}_f,\ t > 0,\ \alpha > 1.$$

It is well defined since $\|y\|^\beta e^{-t\|y\|^\alpha} \in L^\varrho(\mathbb{A}_f)$ for any $\alpha > 1$ and $\beta \geq 0$.

Proposition

$$Z(x,t) = \sum_{q^j < \|x\|^{-1},\ j \neq 0} \Phi\left(q^j\right) \left(e^{-tq^{j\alpha}} - e^{-t(q^j_+)^\alpha}\right) \quad \text{for } t > 0,\ x \in \mathbb{A}_f,$$

Proposition

(i)

$$Z(x,t) \geq 0 \qquad \textit{for } t > 0;$$

(ii)

$$\int_{\mathbb{A}_f} Z(x,t)\, dx_{\mathbb{A}_f} = 1 \quad \textit{for any } t > 0;$$

(iii)

$$Z(x,t) \leq 2t\|x\|^{-\alpha}\Phi\left(\|x\|_{-}^{-1}\right), \quad x \in \mathbb{A}_f \setminus \{0\},\ t > 0.$$

(iv)

$$\int_{\|y\|>\epsilon} Z(y,t)\, dy_{\mathbb{A}_f} \leq C_\epsilon t < +\infty.$$

Theorem

$$\mathcal{T}(t)u = Z(\cdot, t) * u = \int_{\mathbb{A}_f} Z(\cdot - y, t)\, u(y)\, dy_{\mathbb{A}_f}, \quad u \in L_2(\mathbb{A}_f),\ t > 0.$$

Theorem

Let $\alpha > 1$ and let $f \in C^1\left([0, T], L^2(\mathbb{A}_f)\right)$. Then the non-homogeneous Cauchy problem has a unique solution given by

$$u(x,t) = \int_{\mathbb{A}_f} Z(x-y,t)\, u_0(y)\, dy_{\mathbb{A}_f} + \int_0^t \left\{ \int_{\mathbb{A}_f} Z(x-y, t-\tau)\, f(y,\tau)\, dy_{\mathbb{A}_f} \right\} d\tau.$$

We define heat kernel on adeles as

$$Z\left(x,t;\alpha,\beta\right)=Z(x_\infty,t;\beta)Z\left(x_f,t;\alpha\right).$$

Proposition

(i)

$$Z\left(x,t;\alpha,\beta\right)\geq 0 \qquad \textit{for any } t>0;$$

(ii)

$$\int_{\mathbb{A}} Z\left(x,t;\alpha,\beta\right)dx_{\mathbb{A}}=1 \qquad \textit{for any } t>0;$$

(iii)

$$Z\left(x,t;\alpha,\beta\right)*Z\left(x,t';\alpha,\beta\right)=Z\left(x,t+t';\alpha,\beta\right);$$

(iv)

$$\lim_{t\to 0+} Z\left(x,t;\alpha,\beta\right)=\delta\left(x\right) \qquad \textit{in } S'(\mathbb{A}).$$

Semigroup generated by $-D^{\alpha,\beta}$

Theorem

Let $\alpha > 1$ *and* $\beta \in (0, 2]$.

(i) *The operator* $-D^{\alpha,\beta}$ *generates a* C_0 *semigroup* $\left(\mathcal{T}(t;\alpha,\beta)\right)_{t\geq 0}$.

(ii)

$$\mathcal{T}(t;\alpha,\beta)u = Z(\cdot,t) * u.$$

1. S. Torba and W. Zúñiga-Galindo, *Parabolic Type Equations and Markov Stochastic Processes on Adeles*, to appear in Journal of Fourier Analysis and Applications, available at arXiv:1206.5213.

One dimensional p-adic diffusion: