Criteria of measure-preservation for p-adic dynamical systems

(joint talk with Andrei Khrennikov)

Ekaterina Yurova

Linnaeus University, Sweden

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Algebraic and arithmetic dynamics are actively developed fields of general theory of dynamical systems. The bibliography collected by Franco Vivaldi contains 216 articles and books, see

F. Vivaldi, **Algebraic and arithmetic dynamics**,
http://www.maths.qmul.ac.uk/ fv/database/algdyn.pdf

Extended bibliography also can be found in books:

- A. Khrennikov, **Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models.** Kluwer, Dordreht, 1997.
- A. Khrennikov, M. Nilsson, **p-adic deterministic and random dynamics**, Kluwer, Dordrecht, 2004.
- V. Anashin, A. Khrennikov, **Applied Algebraic Dynamics**, de Gruyter Expositions in Mathematics vol 49, Walter de Gruyter (Berlin — New York), 2009.

Definitions

Consider $\langle \mathbb{Z}_p, \mu_p, f \rangle$, where:

- $\mathbb{Z}_p$ is a ring of p-adic integers;
- the normalized Haar measure μ_p;
- $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a μ_p-measurable function that is continuous with respect to p-adic metric.

Definitions

The space $\mathbb{Z}_p$ is equipped with a natural probability measure, namely, the **Haar measure** μ_p normalized so that $\mu_p(\mathbb{Z}_p) = 1$:

Balls $B_{p^{-r}}(a)$ of non-zero radii constitute the base of the corresponding σ-algebra of measurable subsets, $\mu_p(B_{p^{-r}}(a)) = p^{-r}$.
The measure μ_p is a regular Borel measure, so all continuous transformations $f : \mathbb{Z}_p \to \mathbb{Z}_p$ are measurable with respect to μ_p.

A measurable mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called **measure-preserving** if $\mu(f^{-1}(S)) = \mu(S)$ for each measurable subset $S \subset \mathbb{Z}_p$.

Definitions

Let a transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be non-expanding with respect to the p-adic metric; that is, let f be a **1-Lipschitz** with respect to the p-adic metric, i.e. for all $x, y \in \mathbb{Z}_p$

$$|f(x) - f(y)|_p \leq |x - y|_p .$$

The 1-Lipschitz property may be re-stated in terms of congruences. Given $a, b \in \mathbb{Z}_p$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the congruence $a \equiv b \pmod{p^k}$ is well defined: the congruence just means that images of a of b under action of the ring epimorphism $modp^k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$ of the ring $\mathbb{Z}_p$ onto the residue ring $\mathbb{Z}/p^k\mathbb{Z}$ modulo p^k coincide.

The congruence $a \equiv b \pmod{p^k}$ is equivalent to the inequality $|a - b|_p \leq p^{-k}$.

Therefore the transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is **1-Lipschitz** if and only if $f(a) \equiv f(b) \pmod{p^k}$ once $a \equiv b \pmod{p^k}$.

Definitions

A 1-Lipschitz transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called **bijective modulo** p^k if the reduced mapping $f \bmod p^k$ is a permutation on $\mathbb{Z}/p^k\mathbb{Z}$.

The following theorem holds:

Theorem (V. Anashin)

A 1-Lipschitz transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ *is* **measure-preserving** *if and only if it is* **bijective** *modulo* p^k *for all* $k = 1, 2, 3, \ldots$.

Definitions

Given a continuous p-adic function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined on $\mathbb{Z}_p$ and valuated in $\mathbb{Z}_p$. There exists a unique sequence $B_0, B_1, B_2, \ldots$ of p-adic integers such that

$$f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x)$$

for all $x \in \mathbb{Z}_p$, where

$$\chi(m, x) = \begin{cases} 1, & \text{if } |x - m|_p \leq p^{-n} \\ 0, & \text{otherwise} \end{cases}$$

and $n = 1$ if $m = 0$;
n is uniquely defined by the inequality $p^{n-1} \leq m \leq p^n - 1$ otherwise.
This series is called the **van der Put series** of the function f.

Definitions

The number n in the definition of $\chi(m,x)$ has a meaning as

$$\lfloor \log_p m \rfloor = (\text{the number of digits in a base-}p\text{ expansion for } m) - 1,$$

therefore $n = \lfloor \log_p m \rfloor + 1$ for all $m \in \mathbb{N}_0$ and $\lfloor \log_p 0 \rfloor = 0$.
And $\lfloor \alpha \rfloor$ for a real α denotes the nearest to α rational integer which does not exceed α.
Note that $\chi(m,x)$ is merely a **characteristic function of the ball** $\mathbf{B}_{p^{-\lfloor \log_p m \rfloor - 1}}(m) = m + p^{\lfloor \log_p m \rfloor - 1}\mathbb{Z}_p$ of radius $p^{-\lfloor \log_p m \rfloor - 1}$ centered at $m \in \mathbb{N}_0 = \{0, 1, 2 \ldots\}$:

$$\chi(m,x) = \begin{cases} 1, & \text{if } x \equiv m \pmod{p^{\lfloor \log_p m \rfloor + 1}}; \\ 0, & \text{otherwise} \end{cases} =$$

$$\chi(m,x) = \begin{cases} 1, & \text{if } x \in \mathbf{B}_{p^{-\lfloor \log_p m \rfloor - 1}}(m) \\ 0, & \text{otherwise} \end{cases}$$

Definitions

The sequence $B_0, B_1, \ldots B_m$ of **van der Put coefficients** of the function f tends p-adically to 0 as $m \to \infty$, and the series converges uniformly on $\mathbb{Z}_p$, and vice versa.
The coefficients B_m are related to values of the function f as follows.
Let

$$m = m_0 + \ldots + m_{n-2}p^{n-2} + m_{n-1}p^{n-1}$$

be a base-p expansion for m, i.e.,
$m_j \in \{0, \ldots, p-1\}$, $j = 0, 1, \ldots, n-1$ and $m_{n-1} \neq 0$, then

$$B_m = \begin{cases} f(m) - f(m - m_{n-1}p^{n-1}), & \text{if } m \geq p; \\ f(m), & \text{otherwise.} \end{cases}$$

Also note that the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is **1-Lipschitz** iff
$B_m = b_m p^{\lfloor \log_p m \rfloor}$, where $b_m \in \mathbb{Z}_p$.

1-Lipschitz in terms of the coordinate functions

A 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ has the **coordinate representation**:

$$f(x_0+px_1+\ldots+p^kx_k+\ldots) = \varphi_0(x_0)+p\varphi_1(x_0,x_1)+\ldots+p^k\varphi_k(x_0,x_1,\ldots,x_k)+\ldots$$

where $\varphi_k(x_0,x_1,\ldots,x_k)$ are p-valued functions that depend on p-valued variables $x_0,x_1,\ldots,x_k$, $k=0,1,2,\ldots$.

Outline

In this talk

1. Describe all measure-preserving with respect to the measure μ_p 1-Lipschitz functions $f: \mathbb{Z}_p \to \mathbb{Z}_p$ for $p \neq 2$ in terms of
   1.1 **van der Put basis**
   1.2 **coordinate functions**
2. Consider **case** $p = 3$, where all transformations of the field of residues $\mathbb{Z}/3\mathbb{Z}$ can be set as linear polinomials from $\mathbb{Z}/3\mathbb{Z}[x]$, and, in particular, where all bijective transformations can be set via polinomials $ax + b$, $a \neq 0$.

Results: M-P via van der Put basis for $\forall p$

Theorem

Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a 1-Lipschitz function and

$$f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x)$$

be the van der Put representation of this function, where $b_m \in \mathbb{Z}_p, m = 0, 1, 2, \ldots.$
Then f **preserves measure** *iff*

1. $\{b_0, b_1, \ldots, b_{p-1}\}$ *establish a complete set of residues modulo p, i.e. the function f is bijective modulo p;*
2. *for every $m \in \{0, \ldots, p^k - 1\}$ and $k \in \{2, 3, \ldots\}$, the elements in the set*
$$\{b_{m+p^k}, b_{m+2p^k}, \ldots, b_{m+(p-1)p^k}\}$$
are all nonzero residues modulo p.

Results: M-P via van der Put basis for $\forall p$

Second condition of this Theorem means that for $k \geq 1$ and any fixed $\bar{x} = x_0 + px_1 + \ldots + p^{k-1}x_{k-1}$ the functions

$$\beta_{\bar{x}}(h) = \begin{cases} b_{\bar{x}+hp^k}, & \text{if } h = 1, 2, \ldots, p-1 \\ 0, & \text{if } h = 0 \end{cases}$$

are permutations on $\mathbb{Z}/p\mathbb{Z}$. Then one can obtain criterion of measure-preservation for the p-adic functions represented in coordinate form.

Results: M-P via coordinate functions for $\forall p$

Let $\varphi_{k,\bar{x}} \colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be a function obtained from $\varphi_k(x_0, x_1, \ldots, x_k)$ by fixating the values of variables $\bar{x} = (x_0, x_1, \ldots, x_{k-1})$.

Theorem

Let 1-Lipschitz function $f \colon \mathbb{Z}_p \to \mathbb{Z}_p$ has coordinate representation

$$f(x_0+3x_1+\ldots+3^k x_k+\ldots) = \varphi_0(x_0)+p\varphi_1(x_0,x_1)+\ldots+p^k\varphi_k(x_0,x_1,\ldots,x_k)+\ldots$$

where $\varphi_k(x_0, x_1, \ldots, x_k)$ are p-valued functions, $k = 0, 1, 2, \ldots$.
The function f **preserves measure** *iff*

1. *$\varphi_0(x_0)$ is bijective on $\mathbb{Z}/p\mathbb{Z}$;*
2. *$\varphi_{k,\bar{x}} \colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is bijective on $\mathbb{Z}/p\mathbb{Z}$ for any fixed $\bar{x} = (x_0, x_1, \ldots, x_{k-1})$ and $k \geq 1$.*

Results: M-P in additive form for $\forall p$

Theorem

A 1-Lipschitz function $f: \mathbb{Z}_p \to \mathbb{Z}_p$ **preserves measure** *iff it can be represented as*

$$f(x) = \xi(x) + p \cdot h(x),$$

where $h: \mathbb{Z}_p \to \mathbb{Z}_p$ is an arbitrary 1-Lipschitz function, and the functions $\xi(x)$ represented via the van der Put series is such that

$$\xi(x) = \sum_{i=0}^{p-1} G(i)\chi(i,x) + \sum_{k=1}^{\infty} \sum_{m=0}^{p^k-1} \sum_{i=1}^{p-1} g_m(i)p^k \cdot \chi(m + i \cdot p^k, x),$$

where g_m is a permutation on the set $\{1, \ldots, p-1\}$ and G is a permutation on the set $\{0, 1, \ldots, p-1\}$.

Results: M-P via van der Put basis for $p = 3$

Theorem

Let $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ be a 1-Lipschitz function and

$$f(x) = \sum_{m=0}^{\infty} 3^{\lfloor \log_3 m \rfloor} b_m \chi(m, x)$$

be the van der Put representation of this function, where $b_m = b_{\bar{m}+3^n} \in \mathbb{Z}_3,\ m \in \{0, 1, 2, \ldots\}$.
Then f **preserves measure** *iff*

1. $b_m \not\equiv 0 \pmod 3$ *for* $m \geq 3$;
2. $b_{\bar{m}+3^k} + b_{\bar{m}+2\cdot 3^k} \equiv 0 \bmod 3$ *for* $0 \leq \bar{m} \leq 3^k - 1$, $\bar{m} \in \{0, 1, \ldots, 3^k - 1\}$, $k \geq 2$;
3. $(b_0 \bmod 3)$, $(b_1 \bmod 3)$, *and* $(b_2 \bmod 3)$ *establish a complete set of residues modulo* 3, *or in other words,*

$$\begin{cases} b_0 + b_1 + b_2 \equiv 0 \bmod 3 \\ b_0^2 + b_1^2 + b_2^2 \equiv -1 \bmod 3. \end{cases}$$

Results: M-P in additive form for $p = 3$

Theorem

Let $h\colon \mathbb{Z}_3 \to \mathbb{Z}_3$ be an arbitrary 1-Lipschitz function. A 1-Lipschitz function $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ **preserves measure** *iff it can be represented as*

$$f(x) = \xi(x) + 3 \cdot h(x),$$

where

$$\xi(x) = b_0\chi(x,0) + b_1\chi(x,1) + b_2\chi(x,2) + \\ + \sum_{k=1}^{\infty} 3^k \cdot \left(\sum_{\bar{m}=0}^{3^k-1} b_{\bar{m}} \cdot \left(\chi(x, \bar{m} + 3^k) - \chi(x, \bar{m} + 2 \cdot 3^k)\right) \right) + 3\phi(x);$$

and where

1. *for* $b_{\bar{m}} \in \{1, 2\}$
2. *for* $b_0, b_1, b_2 \in \{0, 1, 2\}$ *holds* $b_0 + b_1 + b_2 \equiv 0 \bmod 3$ *and* $b_0^2 + b_1^2 + b_2^2 \equiv -1 \bmod 3$;
3. $\phi(x) = \phi(x_0 + 3x_1 + \ldots + 3^k x_k + \ldots) = \sum_{k=1}^{\infty} 3^k \cdot \frac{x_k(x_k-1)}{2}$.

Results: M-P in additive form for $p = 3$

Set a "fixed" term $\phi(x) = x$.

Theorem

The 1-Lipschitz function $f: \mathbb{Z}_3 \to \mathbb{Z}_3$ **preserves measure** *iff f can be represented as*

$$f(x) = \xi(x) + 3 \cdot h(x),$$

where $h\colon \mathbb{Z}_3 \to \mathbb{Z}_3$ is 1-Lipschitz function and

$$\begin{aligned}\xi(x) =& \xi(x_0 + 3x_1 + \ldots + 3^k x_k + \ldots) = \\ =& b + c \cdot x_0 + x + \sum_{k=1}^{\infty} 3^k \left(I_{M_k}(x_0 + \ldots + 3^{k-1} x_{k-1}) \cdot \frac{x_k \cdot (5 - 3x_k)}{2} \right),\end{aligned}$$

where $c \in \{0, 1\}$, $b \in \{0, 1, 2\}$ and for $M_k \subseteq \{0, 1, \ldots, 3^k - 1\}$

$$I_{M_k}(x_0 + \ldots + 3^{k-1} x_{k-1}) = \begin{cases} 1, & \text{if } x_0 + \ldots + 3^{k-1} x_{k-1} \in M_k \\ 0, & \text{otherwise} \end{cases}$$

(in other words, I_{M_k} is the characteristic function of the set M_k, $k \geq 1$).

Results

From the Theorem above we, in particular, get functions of the form

1. $f(x) = b + x + 3h(x)$ if set $c = 0$ and $M_k = \emptyset$, $k \geq 1$;
2. $f(x) = b + 2x + 3h(x)$ if set $c = 1$ and $M_k = \{0, 1, \ldots, 3^k - 1\}$, $k \geq 1$.

Such classes of 1-Lipschitz measure-preserving functions were obtained by V. Anashin.

Definitions, Ergodicity

A measure-preserving mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is called **ergodic** if $f^{-1}(S) = S$ implies either $\mu_p(S) = 0$ or $\mu_p(S) = 1$.

A 1-Lipschitz transformation $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is called **transitive modulo** p^k if f mod p^k is a permutation that is cycle of length p^k.

The following theorem holds:

Theorem (V. Anashin)

A 1-Lipschitz transformation $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is **ergodic** *if and only if it is* **transitive** *modulo p^k for all $k = 1, 2, 3, \ldots$.*

We obtained **criteria of ergodicity** in the terms of the **coordinate functions** corresponding to the digits in the canonical expansion of p-adic numbers, and presented concrete classes of ergodic functions. A 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ has the **coordinate representation**:

$$f(x_0+px_1+\ldots+p^kx_k+\ldots) = \varphi_0(x_0)+p\varphi_1(x_0,x_1)+\ldots+p^k\varphi_k(x_0,x_1,\ldots,x_k)+\ldots$$

where $\varphi_k(x_0, x_1, \ldots, x_k)$ are p-valued functions that depend on p-valued variables $x_0, x_1, \ldots, x_k$, $k = 0, 1, 2, \ldots$.

Summary of results, Ergodicity for $\forall p$

1. **General criterion**, where ergodicity of the function f is determined via a product of permutations $\varphi_{k,\bar{x}}$ over $\mathbb{Z}/p\mathbb{Z}$ depending on the order of elements in the sequence of residues modulo p^k, where $f_{k-1} = f \pmod{p}^k$ in

$$\tau_k = \left\{\bar{x}, f_{k-1}(\bar{x}), \dots, f_{k-1}^{(p^k-1)}(\bar{x})\right\}.$$

Moreover, conditions of ergodicity does not depend on the choice of the parameter $\bar{x}$, in particular, set $\bar{x} = 0$.

Permutations $\varphi_{k,f_{k-1}^{(i)}(0)}$ can commute, then we can write criterion of ergodicity in "'compact way"'.

Summary of results, Ergodicity for $\forall p$

2. We answered the following question.

Let f be a measure-preserving 1-Lipschitz function.
How much should one change such function to get an ergodic function?

It is enough to set in special way permutation $\varphi_{k,0}$ for arbitrary choosen $\varphi_{k,\bar{x}}$, $\bar{x} \neq 0$ and $k \geq 1$.

3. **Compact description** of some classes of ergodic 1-Lipschitz p-adic functions.
In particular, were described ergodic functions, where

1. $\varphi_{k,\bar{x}} = x_k + \beta(\bar{x})$;
2. $\varphi_{k,\bar{x}} = x_k \cdot \alpha_k(x_0, \ldots, x_s) + \beta(\bar{x})$ for some fixed s.

As Corollary of these results were obtained description of uniformly differentiable modulo p 1-Lipschitz functions, see Open question 4.60, V. Anashin, A. Khrennikov, Applied Algebraic Dynamics.

References

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