Qualitative theory of p-adic dynamical systems

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Contents

Analytic dynamical system. Definition

Analytic dynamical system $= (X, T)$

- X - p-adic analytic manifold
- T - analytic (auto/endo)morphism of X

p-Adic Analytic Manifold

Theorem (J. P. Serre)

Suppose X is compact, non-empty analytic manifold over $\mathbb{Q}_p$ everywhere of the same dimension $d \geq 1$. Then:

- X is the disjoint union of a finite number of balls.
- The number of balls in a decomposition of X into a disjoint union of a finite number of balls is well determined mod $(p-1)$.

Corollary

Such an X is determined, up to an isomorphism, by a number $\tau = \tau(X) \in \{1, 2, \dots, p-1\}$.

Examples

- p-Adic integers $\mathbb{Z}_p$, $\tau\left(\mathbb{Z}_p\right) = 1$
- Projective line $\mathbf{P}^1\left(\mathbb{Q}_p\right)$, $\tau\left(\mathbf{P}^1\right) = 2$
- Unite circle $S = \{x : |x|_p = 1\}$, $\tau(S) = p - 1$

Graph $\Gamma = \Gamma(X)$

- Vertices of Γ are balls in X.
- Two vertices v_1 and v_2 of Γ are connected by an edge $[v_1, v_2]$ iff $v_1 \subset v_2$ and for any ball $v \in X$, $v_1 \subseteq v \subseteq v_2$ we have $v = v_1$ or $v = v_2$.
- $\Gamma(X)$ is a tree.
- Number of connected components of Γ is equal to $\tau(X)$.
- Every component has unique vertice (=root) with p neighbours. Any other vertice has $p + 1$ neighbours.
- There is one-to-one correspondence between points of X and infinite pathes without returns starting at a root point .

Measure

Measure $\mu(B_{root})$ of the root ball B_{root} can be chosen arbitrarily.
$\mu(X) = \sum_{roots} \mu(B_{root}) = 1$.
Measure of a ball at distance n from B_{root} is equal to $p^{-n}\mu(B_{root})$.

Hierarchical dynamical systems

Proposition

Let T be an analytic automorphism of X. That T can be uniquely extended to automorphism of $\Gamma(X)$.

Analytic dynamical system is an example of **hierarchical dynamical system (HDS)**.
Phase space = boundary of a tree.
Dynamical map = boundary trace of a tree endomorphism.

The Poincare recurrence theorem

Theorem 1

Let (X, T) be the measure preserving HDS. By $N_B^n(x)$ denote the number of returns from $x \in B$ into a ball B during the time n. Then we have:

- $N_B^n(x) \to \infty$ when $n \to \infty$ $\forall x, \forall B$;
- $N_B^n(x)$ is indepentent of $x \in B$;

Corollary

HDS is never totally ergodic. It means that $\forall$ HDS (X, T) there exists $k \geq 1$ such that (X, T^k) is not ergodic.

Ergodicity

Theorem 2

The measure-preserving HDS is ergodic iff $\forall B$ there exists the following limit:

$$\lim_{n\to\infty} \frac{1}{n} N_B^n = \mu(B).$$

Theorem 2'

The measure-preserving HDS is ergodic iff $\forall B$ the first return time is equal to $1/\mu(B)$.

Corollary

- For the ergodic measure-preserving HDS there are no periodic orbits.
- Any HDS is uniquely ergodic.

Mixing

Definition

A DS (X, T) is called mixing if $\forall\ A \subset X$ and $\forall\ B \subset X$ the sequence $M_k(A, B) = \mu\left(T^k A \cap B\right)$ tends to $\mu(A)\mu(B)$ as $k \to \infty$.

Theorem 3

The HDS is never mixing.

Hint.

$$M_n(A, B) = \begin{cases} 0, \\ \mu\left(T^n A\right), \\ \mu(B). \end{cases} \quad \not\to \mu(A)\mu(B).$$

Classification of HDS

Proposition

Let (X, T) be the HDS of type 1 ($\tau(X)$=1) then T is an automorphism of X iff T is measure-preserving. In that case T is a permutation of balls of the same measure in every ergodic component. Entropy$(T) = 0$.

Classification of HDS

Theorem 4

Let (X, T) be the HDS of type $\tau = 2$. Then T is the superposition of the following transformations:

- Permutation of roots. Entropy $= 0$.
- n-shift. Entropy $= n$.

Hypothesis

Let (X, T) be the HDS of type $\tau \geq 2$. Then T is the superposition of the following transformations:

- Permutation of roots. Entropy $= 0$.
- $(n_1, n_2, \ldots, n_k)$-shifts, $k = \frac{1}{2}\tau(\tau - 1)$. Entropy - ?.

non-HSD. The baker's map

Let $X = \mathbb{Z}_p \times \mathbb{Z}_p$ with standard metric
$\|z\| = \max\{|x|_p, |y|_p\}, z = (x, y) \in X$.
The baker's map is defined by the formula

$$T(x, y) = \left(\frac{x - x_0}{p}, x_0 + py \right),$$

where $x_0 = x(\mod p)$.

Theorem 6

The baker's map is mixing.

Hint.

$$T\left(B_0 \times B_1\right) = B_0 \times \bigcup_{a=0}^{p-1} B_2(a).$$